On the L^p-theory of the Navier-Stokes equations on Lipschitz domains

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

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aus Wiesbaden

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Tag der Einreichung:	07. Juli 2016
Tag der mündlichen Prüfung:	31. August 2016

Darmstadt, 2017
D17

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Preface

Consider a container Ω filled with a Newtonian fluid. Suppose this fluid is incompressible and has a viscosity ν. Starting to observe the fluid at time $t = 0$, one measures at each point $x \in \Omega$ a velocity $a(x)$, which defines an initial velocity field at each point in the container. Letting the time pass, the fluid motion can be described by means of

$$\text{(NSE)} \qquad \begin{cases} \partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla\pi = 0 & \text{in } \Omega, 0 < t < T \\ \operatorname{div}(u) = 0 & \text{in } \Omega, 0 < t < T \\ u(0) = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, 0 < t < T, \end{cases}$$

where for some $T > 0$, $u : [0, T) \times \Omega \to \mathbb{R}^3$ describes the velocity and $\pi : [0, T) \times \Omega \to \mathbb{R}$ the pressure of the fluid at a given time and location. These equations are called the *Navier-Stokes equations* and appeared for the first time in the first half of the 19th century in the physics literature. In the following, we set $\nu = 1$.

The mathematical foundations were laid in the 1930's in the papers of J. Leray [64–66] and continued by E. Hopf in [51]. In their honor, the class of functions

$$\mathrm{L}^\infty(0, T; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

is named the *Leray-Hopf class*. Here, the subscript σ indicates solenoidal, i.e., divergence-free counterparts of the spaces $\mathrm{L}^2(\Omega; \mathbb{R}^3)$ and $\mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^3)$.

The Leray-Hopf class plays an eminent role in the theory of the Navier-Stokes equations, as the norm of this space represents the energy of u. That is why weak solutions are required to be contained in this space.

Unfortunately, the Leray-Hopf class seems to be too large to deduce uniqueness of weak solutions, as up to the present day, the uniqueness (or non-uniqueness) question for weak solutions in the Leray-Hopf class is unsolved. For a recent article concerning a possible non-uniqueness, see H. Jia and V. Šverák [55].

To reduce the possible number of weak solutions, one can impose further conditions. For example, in the class of all weak solutions u that additionally fulfill the *Serrin condition*

$$u \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega;\mathbb{R}^3)), \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1, \quad p \in (3,\infty), \quad q \in (2,\infty),$$

one can establish uniqueness. Thus, it seems desirable to construct weak solutions in the Leray-Hopf class, that fulfill the Serrin condition. This desire leads to the necessity to set up an L^p-theory for the Navier-Stokes equations with $p \neq 2$.

Another example for the advantages of additional L^p-estimates for weak solutions is given by L. Escauriaza, G. Seregin, and V. Šverák [26]. In the case $\Omega = \mathbb{R}^3$, these authors proved, that weak solutions in the Leray-Hopf class, that additionally lie in the space

$$\mathrm{L}^\infty(0,\infty;\mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^3))$$

are smooth in $[0,\infty) \times \mathbb{R}^3$. This space is often called to be a *critical* space for the Navier-Stokes equations, because the norm of this space is invariant under the natural scaling behavior of solutions to the Navier-Stokes equations.

This is interesting in view of the famous global regularity problem of the Clay Mathematics Institute [32], which calls for a resolution of the question, whether solutions of the Navier-Stokes equations corresponding to initial velocity fields in the Schwartz space are smooth in $[0,\infty) \times \mathbb{R}^3$ and lie in the space $\mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\mathbb{R}^3;\mathbb{R}^3))$.

One cornerstone of an L^p-theory of the Navier-Stokes equations is the development of the semigroup theory for its linearization, i.e., for the

Stokes equations

$$
\text{(SE)} \qquad \begin{cases} \partial_t u - \Delta u + \nabla \pi = 0 & \text{in } \Omega, t > 0 \\ \operatorname{div}(u) = 0 & \text{in } \Omega, t > 0 \\ u(0) = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, t > 0. \end{cases}
$$

The *Stokes semigroup* is then the solution operator $(\mathrm{e}^{-tA})_{t \geq 0}$ to (SE) and the *Stokes operator* A is interpreted to realize the expression "$-\Delta u + \nabla \pi$" in the first equation above.

With this semigroup at hand, the Navier-Stokes equations can be transformed into an integral equation via the variation of constants formula. A breakthrough to tackle the Navier-Stokes equations via this integral equation was performed by T. Kato and H. Fujita in [60]. Even though the work of Kato and Fujita is still in the L^2-setting, it gave rise to the famous work of T. Kato [59], which provides some L^p-theory. In this work, Kato proved the existence of global, so-called, strong solutions to the Navier-Stokes equations in the critical space $\mathrm{L}^\infty(0, \infty; \mathrm{L}^3(\mathbb{R}^3; \mathbb{R}^3))$ if a smallness condition is imposed on the initial datum a. To prove the existence of such a solution, Kato defined an approximation scheme and proved convergence of this scheme by requiring merely the following two ingredients of the Stokes semigroup:

(1) L^p-L^q-estimates of the Stokes semigroup, i.e.,

$$
\|\mathrm{e}^{-tA} f\|_{\mathrm{L}^q(\mathbb{R}^3;\mathbb{R}^3)} \leq C t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p(\mathbb{R}^3;\mathbb{R}^3)} \qquad (\tfrac{3}{2} \leq p \leq q < \infty);
$$

(2) Gradient estimates of the Stokes semigroup, i.e.,

$$
\|\nabla \mathrm{e}^{-tA} f\|_{\mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^9)} \leq C t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^3)}.
$$

Around the same time, Y. Giga proposed a similar iteration scheme in [41], which produces strong solutions in the critical space and works also on domains. This scheme bases upon the validity of L^p-L^q-estimates of the Stokes semigroup for $1 < p \leq q < \infty$ and replaced the gradient estimates by:

(2′) Estimates of the Stokes semigroup applied to a divergence form structure

$$\|\mathrm{e}^{-tA}\mathbb{P}\operatorname{div}(F)\|_{\mathrm{L}^p(\mathbb{R}^3;\mathbb{R}^3)} \leq Ct^{-\frac{1}{2}}\|F\|_{\mathrm{L}^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \qquad (1 < p < \infty).$$

Here, $\mathbb{P}$ denotes the Helmholtz projection, which projects vector fields in $\mathrm{L}^p(\mathbb{R}^3;\mathbb{R}^3)$ onto solenoidal vector fields in $\mathrm{L}^p_\sigma(\mathbb{R}^3)$. To put (2′) into play, Giga used the well-known identity to rewrite the nonlinearity for solenoidal vector fields as

$$\langle u, \nabla\rangle u = \operatorname{div}(u \otimes u),$$

where $u \otimes u$ is the matrix, which arises by multiplying u with the transpose of u. We emphasize that this is a structural information, that Giga exploits in his proof and that is neglected by Kato.

Besides the techniques of Kato and Giga, there is another way to prove the existence of solutions to the Navier-Stokes equations in L^p-spaces; this way requires maximal L^q-regularity of the Stokes operator. Here, one considers the inhomogeneous Stokes equations

$$\text{(ISE)} \qquad \left\{ \begin{aligned} \partial_t u - \Delta u + \nabla \pi &= f && \text{in } \Omega, t > 0 \\ \operatorname{div}(u) &= 0 && \text{in } \Omega, t > 0 \\ u(0) &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, t > 0 \end{aligned} \right.$$

for some $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))$ and asks, whether each of the terms $\partial_t u$ and $Au = -\Delta u + \nabla\pi$ lie in $\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))$. If this is true, then abstract arguments imply that one can estimate

$$\|\partial_t u\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))} + \|Au\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))} \leq C\|f\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))}.$$

Knowing that the Stokes operator has maximal L^q-regularity, one can usually perform a fixed point argument in order to solve the Navier-Stokes equations. The resulting solution then has also the property that

$$\|\partial_t u\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))} + \|Au\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))} < \infty.$$

If A is boundedly invertible, one finds $u \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{R}^3))$. Thus, if p and q can be suitably chosen, one can construct solutions that fulfill the Serrin condition. On bounded, smooth domains and with $q = p$, the first verification of maximal L^q-regularity of the Stokes operator was performed by V. Solonnikov in [91]. A modern approach to the maximal L^q-regularity of the Stokes operator is given by M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis in [35].

In real world applications, however, the container that is filled with the fluid does usually have some edges and corners. Thus, it is natural to assume that Ω is merely a bounded Lipschitz domain. On Lipschitz domains, there are mainly two problems that complicate the analysis:

(i) Second derivatives of $u \in \mathcal{D}(A)$ fail to be L^p-integrable;

(ii) most localization techniques fail due to the lack of smoothness of the boundary.

To tackle the arising problems, techniques from harmonic analysis had to be developed and these lead to the works of E. Fabes, C. Kenig, and G. Verchota [28] dealing with the L^2-Dirichlet problem for the stationary Stokes equations and of Z. Shen [86] dealing with the same problem in the instationary case.

Based on these works P. Deuring and W. von Wahl succeeded to transfer the methods of Kato and Fujita to the case of bounded Lipschitz domains with connected boundary in [22]. This was ultimately improved by M. Mitrea and S. Monniaux in [75], leading to the probably best result, that can be obtained by means of the method of Kato and Fujita on bounded Lipschitz domains. In this context, we mention the work of R. Farwig and H. Sohr [30], still basing upon L^2-theory of the Stokes semigroup, which characterizes the largest possible space of initial values that provides the existence of local strong solutions in Serrin's class $\mathrm{L}^8(0,T;\mathrm{L}^4(\Omega;\mathbb{R}^3))$. The surprising fact here is that Ω is solely assumed to be open and connected.

However, in the end of the 1990s an L^p-theory for the Stokes operator seemed far away. At this time, M. Taylor made the following conjecture in [93]:

> *For a given Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the Stokes operator generates an analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$, provided that $3/2 - \varepsilon < p < 3 + \varepsilon$.*

Shortly after this conjecture was made, P. Deuring showed in [21] that this interval of p's cannot be larger. In other words, he showed that for a given $p \notin [3/2, 3]$, there exists a bounded Lipschitz domain Ω, such that the Stokes operator does not generate a strongly continuous semigroup on $\mathrm{L}^p_\sigma(\Omega)$. It lasted twelve years until Z. Shen could resolve Taylor's conjecture affirmatively in his seminal paper [89].

Besides optimal embeddings of the domain of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ into certain Bessel potential spaces derived by M. Mitrea and M. Wright in [77], these results seem to be everything that is known about the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ in the case of Lipschitz domains and Dirichlet boundary conditions. Note that for other, so-called "free boundary" conditions, semigroup theory in L^p and theory for the solvability in the critical space is known due to M. Mitrea and S. Monniaux [73,74]. However, for the Navier-Stokes equations on bounded Lipschitz domains subject to Dirichlet boundary conditions there seems to be no existing L^p-theory and this sets the starting point for this thesis.

Overview

The main objective of this thesis is to prove the existence of solutions to the Navier-Stokes equations in the critical space $\mathrm{L}^\infty(0,\infty;\mathrm{L}^3(\Omega;\mathbb{R}^3))$ on bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$. In this context, we establish L^p-L^q-estimates of the Stokes semigroup for $3/2 - \varepsilon < p \leq q < 3 + \varepsilon$ in Theorem 5.2.22. Furthermore, in the same theorem, we obtain a weaker form of the gradient estimates in L^3, namely

$$\|\nabla \mathrm{e}^{-tA} f\|_{\mathrm{L}^q(\Omega;\mathbb{R}^9)} \leq C t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{R}^3)}, \tag{$2''$}$$

whenever $3/2 - \varepsilon < p \leq q < 3$ with $p \leq 2$. While this type of estimate obstructs the way to follow Kato's proof for the convergence of the iteration scheme, the hope is not lost in view of Giga's proof. As it was mentioned above, Giga exploits the special structure of the nonlinearity of the Navier-Stokes equations. Peculiarly, this is not reflected in the estimates he requires, as $(2')$ is simply the dual version of (2). In Section 6.3, we review his proof and show that $(2')$ can be weakened to the dual version of $(2'')$. This leads to the following main result:

> **Theorem.** *For all $a \in \mathrm{L}^3_\sigma(\Omega)$ there exists a time $T_0 > 0$ such that there exists a mild solution u to* (NSE) *on $[0, T_0)$ with initial datum a and $u \in \mathrm{L}^\infty(0, T_0; \mathrm{L}^3(\Omega; \mathbb{R}^3))$. Furthermore, if a is small in the $\mathrm{L}^3(\Omega; \mathbb{R}^3)$-norm, this solution is global, i.e., $T_0 = \infty$.*

Pursuing an L^p-theory of the Navier-Stokes equations on bounded Lipschitz domains in three or more dimensions d via maximal L^q-regularity, the first step is done in Theorem 5.2.24:

> **Theorem.** *There exists $\varepsilon > 0$ such that for all*
>
> $$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$
>
> *the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ has maximal L^q-regularity for $1 < q < \infty$.*

Combining this result with the above mentioned embeddings of the domain of the Stokes operator into Bessel potential spaces and other estimates proven by R. Brown and Z. Shen in [13], reveals in three dimensions the following existence theorem, which is Theorem 6.1.3 of this thesis:

> **Theorem.** *There is $\varepsilon > 0$ such that for all $2 \leq p < 3 + \varepsilon$ there is $q(p, \varepsilon) \geq 2$ such that for all $q(p, \varepsilon) < q < \infty$ and all initial data in the real interpolation space $(\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A))_{1-1/q,q}$ that are small enough, there exists a global strong solution u to* (NSE)*. If $p = 3$ and q is large enough, the solution u lies in the critical space $\mathrm{L}^\infty(0, \infty; \mathrm{L}^3(\Omega; \mathbb{R}^3))$ and for p close to 3, q can be chosen such that u satisfies Serrin's condition.*

In order to prove the maximal L^q-regularity of the Stokes operator in $\mathrm{L}^p_\sigma(\Omega)$, we appeal to the characterization of maximal L^q-regularity via square function estimates. As the square function estimates for $p = 2$ are immediate, we proceed by extrapolating their validity from $p = 2$ to $p > 2$. This extrapolation bases upon a Banach space valued version of the L^p-extrapolation theorem of Shen proven in [87]. We review his scalar-valued proof in Chapter 3, extend it to the Banach space valued setting, and weaken the required geometrical assumption of bounded Lipschitz domains to mere open sets. The proof of the maximal L^q-regularity of the Stokes operator in $\mathrm{L}^p_\sigma(\Omega)$ is then an imitation of SHEN'S proof of the resolvent estimates in the Banach space valued setting.

In Chapter 7, we show that this way of proving maximal L^q-regularity does not only work for the Stokes operator, but also for higher-order elliptic systems in divergence form with L^∞-coefficients. Here, the ellipticity is enforced by a Gårding type inequality and the operator is complemented with homogeneous mixed Dirichlet/Neumann boundary conditions. The geometric setup is far more general than the one of bounded Lipschitz domains. In the case of pure Dirichlet boundary conditions, for example, the underlying set is simply assumed to be open. This result is presented in Theorem 7.2.4.

As it was mentioned above, we did not succeed to establish the gradient estimates of the Stokes semigroup in L^3. However, in Section 5.3, we present two possible approaches to these estimates. The first approach leads to the fact, that the boundedness of the H^∞-calculus of the Stokes operator is sufficient for the validity of the gradient estimates. Note that on bounded Lipschitz domains, this is not a trivial fact, because the domain of the Stokes operator does not coincide with a Sobolev space of second order. The second approach imitates a proof of the gradient estimates in the elliptic situation. The crucial ingredient in this elliptic situation is an estimate arising from the regularity theory of the L^2-Dirichlet problem of the resolvent equation of the elliptic system at stake. We establish this regularity theory for the L^2-Dirichlet problem of the Stokes resolvent in Chapter 4. Unfortunately, the estimate arising in this resolution contains an additional term in comparison to the elliptic estimate. This is the same term, which complicated the proof of the resolvent estimates of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$. The discussion on the gradient estimates closes with a proof of the fact, that the imitation always fails if the additional term is estimated in the most obvious way.

In Section 5.1, we deal with the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for $d \geq 2$. Even though, this is already well-investigated by E. FABES, O. MENDEZ, and M. MITREA [29] in three or more dimensions and by D. MITREA [72] in two dimensions, we provide a different approach, which delivers weaker results in $d \geq 4$ dimensions and the same results in two and three dimensions. Moreover, our approach enables us, to consider also special Lipschitz domains, i.e., domains above the graph of a Lipschitz continuous function on $\mathbb{R}^{d-1}$. This includes for example two dimensional domains that resemble a sector.

Finally, Chapter 1 and Chapter 2 do not contain new results. The first chapter starts with an introduction to the required function spaces and closes with a detailed overview over first-order Sobolev spaces on the boundary of a Lipschitz domain. We spend much effort on this overview, because the author felt that it is hard to find proofs of some of the fundamental facts concerning these matters. The second chapter introduces to the basic operator theoretical notions and to the abstract theory of maximal L^q-regularity that is needed later on.

Acknowledgement

First of all I would like to warmly thank my adviser Robert Haller-Dintelmann for his guidance and his counsel during my time as a PhD student and also the years before that as an undergraduate and graduate student. He always had an open door and a good advice for any kind of question. I am thankful to Matthias Hieber who made this dissertation possible by offering me a position during the time I was waiting on my scholarship and I thank him for many good suggestions to all kinds of problems.

I would like to thank Russell Brown for his hospitality during my stay at the University of Kentucky and for answering any of my questions even afterwards my stay. Moreover, I thank Jürgen Saal and Zhongwei Shen for some valuable remarks and their interest in my work. I thank Gregory Verchota for sending me a hardcopy of the appendix of his PhD thesis via mail.

I am indebted to Moritz Egert for spending hours and days for interesting and helpful discussions and for many good suggestions concerning my script.

In the math department in Darmstadt, I thank Esther Bauer, Martin Bolkart, Aday Celik, Mathis Gries, Amru Hussein, David Meffert, Jens-Henning Möller, Martin Saal, Jonas Sauer, Katharina Schade, Anton Seyfert, Verena Soga, Fabian Völz, and David Wegmann for the great time and Martin Bolkart, Moritz Egert, Amru Hussein, Martin Saal, Jonas Sauer, Anton Seyfert, and David Wegmann for their valuable proofreading.

I am deeply grateful to my girlfriend Fabienne Blum, to my family

André, Bruno, Helga, Julia, Mirjam Tolksdorf, Dustin Finger, and Carina, Carmen, and Hans-Joachim Blum. This dissertation would not have been possible without their support.

Finally, I thank my friends Ferdinand Cloos, Kathrin Ehrler, Jens Hartwig, Felix Heinrich, Max Karl, Sonja Kugler, André Röder, and Henrike Zimmer for the great time we shared together.

My work was supported by the Studienstiftung des deutschen Volkes.

Thank you!

Zusammenfassung in deutscher Sprache

Der Hauptgegenstand dieser Dissertation sind die inkompressiblen Navier-Stokes-Gleichungen auf einem beschränkten Lipschitz-Gebiet Ω im dreidimensionalen euklidischen Raum. Diese sind gegeben durch das folgende System partieller Differentialgleichungen

$$\text{(NSG)} \qquad \left\{ \begin{aligned} \partial_t u - \nu \Delta u + \langle u, \nabla \rangle u + \nabla \pi &= 0 && \text{in } \Omega, 0 < t < T \\ \operatorname{div}(u) &= 0 && \text{in } \Omega, 0 < t < T \\ u(0) &= a && \text{in } \Omega \\ u &= 0 && \text{auf } \partial\Omega, 0 < t < T \end{aligned} \right.$$

und beschreiben das Geschwindigkeitsfeld $u : [0, T) \times \Omega \to \mathbb{R}^3$ eines Fluids und den Druck $\pi : [0, T) \times \Omega \to \mathbb{R}$, der in dem Fluid herrscht. Die weiteren Größen in (NSG) sind eine Zeit $T > 0$, welche den Endzeitpunkt darstellt, bis wann das Fluid betrachtet werden soll, und a beschreibt das Geschwindigkeitsfeld, welches zum Anfangszeitpunkt der Betrachtungen in Ω vorherrscht. Der Einfachheit halber wird im Folgenden die kinematische Viskosität ν auf Eins gesetzt.

Das Ziel dieser Untersuchung ist die Übertragung zweier klassischer Zugänge zur L^p-Theorie auf dem Ganzraum beziehungsweise auf beschränkten Gebieten mit glattem Rand auf beschränkte Lipschitz-Gebiete. Der erste Zugang entspricht dem von T. Kato [59] und Y. Giga [41] über die Existenz von starken Lösungen im kritischen Raum

$$\mathrm{L}^\infty(0, T; \mathrm{L}^3(\mathbb{R}^3; \mathbb{R}^3)).$$

Der zweite Zugang entspricht dem von V. Solonnikov um über die sogenannte maximale L^q-Regularität an die Existenz von Lösungen von (NSG) zu gelangen. Das erste Hauptresultat dieser Arbeit ist die Verallgemeinerung der Resultate von Kato und Giga und ist im wesentlichen durch den folgenden Hauptsatz widergegeben:

> **Hauptsatz.** *Für alle* $a \in \mathrm{L}^3_\sigma(\Omega)$ *existiert eine Zeit* $T_0 > 0$*, sodass eine milde Lösung* u *von* (NSG) *auf dem Intervall* $[0, T_0)$ *mit Anfangsgeschwindigkeit* a *und* $u \in \mathrm{L}^\infty(0, T_0; \mathrm{L}^3(\Omega; \mathbb{R}^3))$ *existiert. Ist die* L^3*-Norm von* a *klein genug, so existiert die Lösung sogar global in der Zeit, d.h.* $T_0 = \infty$*.*

Dieser Hauptsatz ist in der Arbeit unter Theorem 6.1.6 zu finden. Weiterhin steht der Index σ bei $\mathrm{L}^3_\sigma(\Omega)$ für „divergenzfrei“, d.h. dies ist der abgeschlossene Untervektorraum von $\mathrm{L}^3(\Omega; \mathbb{C}^3)$ der divergenzfreien Vektorfelder auf Ω.

Um eine L^p-Theorie der Navier-Stokes-Gleichungen über den Zugang der maximalen L^q-Theorie zu erlangen, haben wir in Theorem 5.2.24 auf drei- und höherdimensionalen Lipschitz-Gebieten Ω folgenden Hauptsatz bewiesen:

> **Hauptsatz.** *Es existiert eine Konstante* $\varepsilon > 0$*, sodass für alle*
>
> $$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$
>
> *der Stokes-Operator* A *auf* $\mathrm{L}^p_\sigma(\Omega)$ *für alle* $1 < q < \infty$ *die Eigenschaft der maximalen* L^q*-regularität besitzt.*

Basierend auf diesem Resultat ist es auf dreidimensionalen, beschränkten Lipschitz-Gebieten möglich eine Fixpunktiteration durchzuführen, falls der Definitionsbereich des Stokes-Operators auf $\mathrm{L}^p_\sigma(\Omega)$ in einen Bessel-Potential-Raum mit einer Differenzierbarkeit größer als Eins einbettet und falls bestimmte L^∞-Abschätzungen für den Stokes-Operator auf $\mathrm{L}^2_\sigma(\Omega)$ gelten. Das Erstgenannte ist ein Ergebnis von M. Mitrea und M. Wright in [77] und das Zweitgenannte von R. Brown und Z. Shen in [13]. Dies führt zu folgendem Hauptsatz, der in der Arbeit unter Theorem 6.1.3 zu finden ist:

Hauptsatz. *Es existiert eine Konstante* $\varepsilon > 0$*, sodass für alle* $2 \leq p < 3 + \varepsilon$ *eine Konstante* $q(p, \varepsilon) \geq 2$ *existiert, sodass für alle* $q(p, \varepsilon) < q < \infty$ *und alle Anfangsgeschwindigkeiten* a*, dessen Norm bezüglich der Norm des reellen Interpolations-Raumes* $(\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A))_{1-1/q,q}$ *klein genug ist, eine globale starke Lösung von* (NSG) *existiert. Ist im Falle* $p = 3$ *die Zahl* q *groß genug, so liegt diese Lösung in* $\mathrm{L}^\infty(0, \infty; \mathrm{L}^3(\Omega; \mathbb{R}^3))$*. Ist* $p > 3$ *nahe genug an* 3*, so erfüllt diese Lösung die sogenannte Serrin-Bedingung.*

Um die Eigenschaft der maximalen L^q-Regularität des Stokes-Operators auf $\mathrm{L}^p_\sigma(\Omega)$ zu beweisen, nutzen wir die Charakterisierung der maximalen L^q-Regularität über sogenannte „square function estimates“. Diese sind im Falle $p = 2$ eine einfache Folgerung der Resolventenabschätzungen des Stokes-Operators auf $\mathrm{L}^2_\sigma(\Omega)$. Um die Gültigkeit dieser „square function estimates“ für $p > 2$ zu erhalten, sollen diese von $p = 2$ auf $p > 2$ extrapoliert werden. Hierfür wird eine Banachraumwertige Version des L^p-Extrapolationssatzes von Z. Shen, welcher im skalarwertigen Fall durch Shen in [87] bewiesen wurde, benötigt. In Kapitel 3 wird diese Banachraumwertige Version bewiesen und dabei werden jegliche geometrischen Voraussetzungen, welche von Shen benötigt wurden, abgeschwächt, sodass dieses Resultat nun auf beliebigen offenen Mengen gültig ist. Um die Voraussetzungen dieses Banachraumwertigen L^p-Extrapolationssatzes für den Stokes-Operator auf $\mathrm{L}^p_\sigma(\Omega)$ zu verifizieren, wird Shen's Beweis der Resolventenabschätzungen des Stokes-Operators auf $\mathrm{L}^p_\sigma(\Omega)$, welcher in [89] geführt wurde, auf die Banachraumwertige Situation angepasst.

In Kapitel 7 zeigen wir, dass diese Art und Weise, um maximale L^q-Regularität zu beweisen, auch für elliptische Systeme höherer Ordnung in Divergenzform mit L^∞-Koeffizienten funktioniert. Es werden solche Systeme betrachtet, welche die Gårding'sche Ungleichung erfüllen. Weiterhin wird der Operator mit homogenen, gemischten Dirichlet/Neumann-Randbedingungen komplementiert. Die Geometrie des Gebietes hängt von der Wahl der Randbedingungen ab und ist um einiges allgemeiner als die von beschränkten Lipschitz-Gebieten. Liegen zum Beispiel reine Dirichlet-Randbedingungen vor, so reicht es aus, wenn Ω eine beschränkte offene Menge ist. Dieses Resultat ist Theorem 7.2.4 dieser Arbeit.

Eine wichtige Zutat um Kato's Zugang für die L^p-Theorie der Navier-Stokes-Gleichungen zu folgen, sind die sogenannten Gradientenabschätzungen der Stokes-Halbgruppe. Genauer gesagt, werden diese Abschätzungen in $\mathrm{L}^3(\Omega;\mathbb{R}^3)$ benötigt. Leider konnten diese im Verlaufe dieser Dissertation nicht bewiesen werden. Dennoch können wir zwei interessante Folgerungen aus zwei möglichen Ansätzen für einen Beweis der Gradientenabschätzungen gewinnen, welche in Unterkapitel 5.3 diskutiert werden. Aus dem ersten Ansatz gewinnen wir das Resultat, dass die Beschränktheit des H^∞-Kalküls des Stokes-Operators genügt um die Gradientenabschätzungen zu beweisen. Wir weisen darauf hin, dass dies nicht trivial ist, da der Definitionsbereich des Stokes-Operators keinem Sobolev-Raum zweiter Ordnung entspricht. Im zweiten Ansatz wird ein Vergleich zu einem Beweis der Gradientenabschätzungen für die Halbgruppe, welche von einem elliptischen System zweiter Ordnung mit konstanten Koeffizienten erzeugt wird, gezogen. Die essentielle Zutat für diesen Beweis sind Abschätzungen, die in der Regularitätstheorie des L^2-Dirichlet Problems des Resolventenproblems ebendieser elliptischen Systeme auftreten. Diese Regularitätstheorie wird in Kapitel 4 für die Stokes-Resolventen-Gleichung entwickelt. Im Verlgeich zur elliptischen Situation tritt jedoch ein weiterer Term in der eben erwähnten essentiellen Abschätzung auf. In der Diskussion in Unterkapitel 5.3 beweisen wir schließlich, dass ein Imitat des Beweises der elliptischen Situation immer zu einem Widerspruch führt, wenn der zusätzliche Term auf die offensichtlichste Art und Weise abgeschätzt wird.

In Unterkapitel 5.1 studieren wir für $d \geq 2$ die Helmholtz-Projektion auf $\mathrm{L}^p(\Omega;\mathbb{C}^d)$. In drei und mehr Raumdimensionen wurde diese Projektion in [29] bereits vollständig durch E. Fabes, O. Mendez und M. Mitrea und in zwei Raumdimensionen durch D. Mitrea in [72] untersucht. Wir werden einen alternativen Ansatz vorstellen, welcher im Vergleich zu obigen Ergebnissen für $d \geq 4$ schwächere Resultate und für $d = 2$ und $d = 3$ das selbe Resultat liefert. Weiterhin kann der hier vorgestellte Ansatz auch auf eine bestimmte Klasse von unbeschränkten Lipschitz-Gebieten angewendet werden. Diese sind solche Gebiete, welche oberhalb des Graphs einer auf $\mathbb{R}^{d-1}$ Lipschitz stetigen Funktion liegen. Diese Klasse von Gebieten beinhaltet zum Beispiel in zwei Dimensionen sektorähnliche Gebiete.

Die ersten beiden Kapitel dieser Arbeit enthalten keine neuen Resultate. Das erste Kapitel dient größtenteils der Begriffsbildung und beginnt mit einer Einführung in diejenigen Funktionenräume, die in dieser Arbeit benötigt werden. Weiterhin geben wir eine detaillierte Einführung über Sobolev-Räume erster Ordnung auf dem Rand eines beschränkten Lipschitz-Gebietes, da hier dem Autor aufgefallen ist, dass diese Räume eine häufige Verwendung finden, jedoch stets sehr knapp eingeführt werden. Im zweiten Kapitel werden die operatortheoretischen Grundlagen der Arbeit geschaffen. Es umfasst einen kurzen Überblick über Funktionalkalküle und analytische Halbgruppen sowie eine Einführung in die abstrakte Theorie der maximalen L^q-Regularität.

CHAPTER 1

Preliminaries in the theory of function spaces

In this chapter, we will introduce the basic function spaces used in this treatise and discuss their relevant properties. We will start by introducing the Lebesgue, Sobolev, and Bessel potential spaces, as well as their vector-valued analogues. For further reading consult the books of Yosida [101], Triebel [94], and Hytönen, van Neerven, Veraar, and Weis [52]. Afterwards, we give an introduction to basic properties of the distribution function of a function f and to weak Lebesgue spaces. Here, we rely on Grafakos' book [44]. As solenoidal analogues of some of the above mentioned spaces play an eminent role in the study of fluid mechanics, we include a short introduction to these spaces in the special case, where the underlying domain is a bounded Lipschitz domain. These spaces were thoroughly investigated by Fabes, Mendez, and M. Mitrea [29] and by D. Mitrea [72]. Thereafter, we make a small detour via the complex and real interpolation methods following the introduction of Lunardi [67].

In the theory of the Laplacian and the Stokes system on Lipschitz domains notions like the non-tangential maximal function and non-tangential convergence, as well as Sobolev spaces on the boundaries of Lipschitz domains are fundamental. These are introduced in Section 1.3 which fills the details into Verchota's exposition [95]. This section is particularly detailed because in the existing literature introductions to Sobolev spaces on the boundaries of Lipschitz domains are rather short. However, note that the work of Mitrea and Wright [77] contains a short but nice introduction to these spaces.

1.1 Function spaces

1.1.1 Bochner-Lebesgue spaces

Let X be a real or complex Banach space and $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Proceeding as Yosida in [101, Sec. V.5] one can introduce the *Bochner integral*. A measurable function $f : \Omega \to X$ is then called *Bochner integrable* or simply integrable if the real-valued function $\omega \mapsto \|f(\omega)\|_X$ is integrable in the sense on Lebesgue. We will write

$$\int_\Omega f \, \mathrm{d}\mu$$

for the integral of f. If $\mu(\Omega)$ is finite and not zero, we will sometimes write

$$⨍_\Omega f \, \mathrm{d}\mu := \frac{1}{\mu(\Omega)} \int_\Omega f \, \mathrm{d}\mu$$

for the *average of f over* Ω. The L^p-spaces are introduced in the following definition.

Definition 1.1.1. For $1 \leq p \leq \infty$ let $\mathrm{L}^p(\Omega, \mu; X)$ denote the space of all equivalence classes of measurable functions which coincide μ-a.e. and whose L^p-norm

$$\begin{aligned} \|f\|_{\mathrm{L}^p(\Omega,\mu;X)} &:= \Big(\int_\Omega \|f\|_X^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} \qquad (1 \leq p < \infty), \\ \|f\|_{\mathrm{L}^\infty(\Omega,\mu;X)} &:= \inf\{\alpha \geq 0 : \mu(\{\|f\|_X > \alpha\}) = 0\} \end{aligned}$$

is finite. If $X = \mathbb{R}$ or $X = \mathbb{C}$, we write $\mathrm{L}^p(\Omega, \mu)$ for $\mathrm{L}^p(\Omega, \mu; X)$.

Definition 1.1.2. If $\Omega \subset \mathbb{R}^d$ is a topological space and μ a measure, whose σ-algebra contains the Borel sets on Ω, we write $\mathrm{L}^p_{\mathrm{loc}}(\Omega, \mu; X)$ for the set of all functions whose restrictions to K are in $\mathrm{L}^p(K, \mu; X)$ for every compact set $K \subset \Omega$.

Convention 1.1.3. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$ will be denoted by $|A|$ and its volume element is denoted by $\mathrm{d}x$. If Ω is endowed with the Lebesgue measure, we write $\mathrm{L}^p(\Omega; X)$ for $\mathrm{L}^p(\Omega, \mu; X)$. The same applies to the spaces of locally L^p-integrable functions and to all upcoming function spaces.

The following proposition establishes Young's inequality for convolutions if one function is defined only on a subset Ω of $\mathbb{R}^d$.

Proposition 1.1.4. *Let $\Omega \subset \mathbb{R}^d$, μ be a σ-finite measure on Ω, and $1 \leq p < \infty$. Let $g : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a function such that the function defined by $\Omega \times \Omega \ni (x, y) \mapsto g(x - y)$ is measurable with respect to the product measure $\mu \times \mu$ and such that*

$$A + B := \sup_{x \in \Omega} \|g(x - \cdot)\|_{\mathrm{L}^1(\Omega,\mu)} + \sup_{y \in \Omega} \|g(\cdot - y)\|_{\mathrm{L}^1(\Omega,\mu)} < \infty.$$

If $f \in \mathrm{L}^p(\Omega, \mu)$, then $x \mapsto \int_\Omega g(x - y) f(y) \, \mathrm{d}\mu(y) \in \mathrm{L}^p(\Omega, \mu)$ and

$$\Big\| \int_\Omega g(\cdot - y) f(y) \, \mathrm{d}\mu(y) \Big\|_{\mathrm{L}^p(\Omega,\mu)} \leq A^{1 - \frac{1}{p}} B^{\frac{1}{p}} \|f\|_{\mathrm{L}^p(\Omega,\mu)}.$$

Proof. Let first $p = 1$. By virtue of Tonelli's theorem for σ-finite measure spaces, see Billingsley [10, Thm. 18.3], the function

$$\Omega \ni x \mapsto \int_\Omega |g(x - y)| \, |f(y)| \, \mathrm{d}\mu(y) \tag{1.1}$$

is measurable. The estimate directly follows from Tonelli's theorem; Fubini's theorem [10, Thm. 18.3] shows that (1.1) is measurable without the absolute value inside the integral.

Next, let $p > 1$ and let f be a simple function. An application of Hölder's inequality yields

$$\Big| \int_\Omega g(x - y) f(y) \, \mathrm{d}\mu(y) \Big| \leq A^{\frac{1}{p'}} \Big(\int_\Omega |g(x - y)| \, |f(y)|^p \, \mathrm{d}\mu(y) \Big)^{\frac{1}{p}},$$

where p' is the Hölder conjugate exponent of p. The convolution integral is measurable in x as simple functions lie in $\mathrm{L}^1(\Omega)$. Taking the pth power of this inequality, integrating over Ω, and applying Tonelli's theorem gives

$$\Big\| \int_\Omega g(\cdot - y) f(y) \, \mathrm{d}\mu(y) \Big\|^p_{\mathrm{L}^p(\Omega,\mu)} \leq A^{p-1} \Big\| \int_\Omega |g(x - \cdot)| \, |f(\cdot)|^p \, \mathrm{d}\mu(x) \Big\|_{\mathrm{L}^1(\Omega,\mu)}.$$

This establishes the estimate for simple functions. As simple functions are dense in $\mathrm{L}^p(\Omega)$ the estimate follows for all L^p-functions and thus also the measurability of (1.1) without the absolute value inside the integral. $\square$

Remark 1.1.5. Proposition 1.1.4 is often used in the theory of layer potentials with Ω being the boundary of a bounded Lipschitz domain, μ being the surface measure, and g a function that is closely related to a fundamental solution of the given partial differential equation. However, we have not found a proof of Proposition 1.1.4 and therefore decided to present the proof here.

1.1.2 Sobolev and Bessel potential spaces on $\mathbb{R}^d$

Let X be a complex Banach space and let $\mathbb{R}^d$ be endowed with the Lebesgue measure. In what follows, the *space of X-valued Schwartz functions* $\mathcal{S}(\mathbb{R}^d; X)$ is defined via the usual family of seminorms and the *space of X-valued tempered distributions* as $\mathcal{S}'(\mathbb{R}^d; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d; \mathbb{C}), X)$ (the space of bounded linear operators from $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$ into X), see Amann [3, Subsec. III.4.1] for a detailed introduction.

Definition 1.1.6. For $1 \leq p < \infty$ and $k \in \mathbb{N}$, define the *Sobolev space* $\mathrm{W}^{k,p}(\mathbb{R}^d; X)$ as the space of all regular distributions $u \in \mathcal{S}'(\mathbb{R}^d; X)$ such that $\partial^\alpha u \in \mathrm{L}^p(\mathbb{R}^d; X)$ for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ endowed with the norm

$$\|u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d;X)} := \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^d;X)}^p \Big)^{\frac{1}{p}}.$$

Here, for a multi-index $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha u$ denotes $\partial_{\alpha_1} \dots \partial_{\alpha_d} u$.

In order to introduce the Bessel potential spaces denote the Fourier transform on $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$ by $\mathfrak{F}$ and define the Fourier transform of an element in $\mathcal{S}'(\mathbb{R}^d; X)$ as usual, compare Amann [3, Subsec. III.4.2].

Definition 1.1.7. Let $1 \leq p < \infty$ and $0 \leq s < \infty$. The *Bessel potential space* $\mathrm{H}^{s,p}(\mathbb{R}^d; X)$ is defined as the space of all regular distributions $u \in \mathcal{S}'(\mathbb{R}^d; X)$ such that the *Bessel potential norm*

$$\|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^d;X)} := \|\mathfrak{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathfrak{F} u\|_{\mathrm{L}^p(\mathbb{R}^d;X)}$$

is finite.

Remark 1.1.8. It is clear that $\mathrm{H}^{0,p}(\mathbb{R}^d;X) = \mathrm{L}^p(\mathbb{R}^d;X)$. The connection between $\mathrm{W}^{k,p}(\mathbb{R}^d;X)$ and $\mathrm{H}^{s,p}(\mathbb{R}^d;X)$ is discussed in Section 1.2, when the notion of Banach spaces of class $\mathcal{HT}$ is introduced.

Let $\Omega \subset \mathbb{R}^d$ be an open set. The counterparts of the Sobolev and Bessel potential spaces on Ω are defined as follows.

Definition 1.1.9. Let $1 \le p < \infty$, $k \in \mathbb{N}$, and $0 \le s < \infty$. Then the *Sobolev space* $\mathrm{W}^{k,p}(\Omega;X)$ and the *Bessel potential space* $\mathrm{H}^{s,p}(\Omega;\mathrm{X})$ are defined as the sets of all restrictions to Ω of elements in $\mathrm{W}^{k,p}(\mathbb{R}^d;X)$ and $\mathrm{H}^{s,p}(\mathbb{R}^d;X)$, respectively. They are endowed with the quotient norm

$$\|u\|_{\mathrm{W}^{k,p}(\Omega;X)} := \inf\{\|v\|_{\mathrm{W}^{k,p}(\mathbb{R}^d;X)} : v \in \mathrm{W}^{k,p}(\mathbb{R}^d;X) \text{ and } v|_\Omega = u\}$$
$$\|u\|_{\mathrm{H}^{s,p}(\Omega;X)} := \inf\{\|v\|_{\mathrm{H}^{s,p}(\mathbb{R}^d;X)} : v \in \mathrm{H}^{s,p}(\mathbb{R}^d;X) \text{ and } v|_\Omega = u\}.$$

The *local Sobolev spaces* $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega;X)$ are defined as the set of functions whose derivatives up to order k lie in $\mathrm{L}^p_{\mathrm{loc}}(\Omega;X)$.

Remark 1.1.10. Another common way to define Sobolev spaces on open sets is to say that $\mathcal{W}^{k,p}(\Omega;X)$ is the space of functions $u \in \mathrm{L}^p(\Omega;X)$ such that the distributional derivatives $\partial^\alpha u$ lie in $\mathrm{L}^p(\Omega;X)$ for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$, endowed with the norm

$$\|u\|_{\mathcal{W}^{k,p}(\Omega;X)} := \Big(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^p_{\mathrm{L}^p(\Omega;X)}\Big)^{\frac{1}{p}}. \tag{1.2}$$

If there is a Sobolev extension operator $E : \mathcal{W}^{k,p}(\Omega;X) \to \mathrm{W}^{k,p}(\mathbb{R}^d;X)$, i.e., a bounded operator E who is a right inverse of the operator that restricts functions to Ω, then we have $\mathcal{W}^{k,p}(\Omega;X) = \mathrm{W}^{k,p}(\Omega;X)$ with equivalent norms. This applies for instance if Ω is a bounded Lipschitz domain and $X = \mathbb{C}^N$, $N \in \mathbb{N}$, see Stein [92, Thm. VI.5]. Here, we say that Ω is a bounded Lipschitz domain if Ω is a bounded domain, whose boundary is locally represented as the graph of a Lipschitz continuous function, cf. Definition 1.3.1.

In many occasions there appear vector fields, whose components vanish on some closed portions $D_1, \dots, D_N$ of the boundary. Let us define Sobolev spaces adapted to this special situation.

Definition 1.1.11. Let $D \subset \partial\Omega$ be closed and

$$\mathrm{C}_D^\infty(\Omega) := \{\varphi|_\Omega : \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp}(\varphi) \cap D = \emptyset\},$$

where $\mathrm{C}_c^\infty(\mathbb{R}^d)$ stands for all compactly supported, smooth functions on $\mathbb{R}^d$.

For $1 \leq p < \infty$ and $k \in \mathbb{N}$ define the space $\mathrm{W}_D^{k,p}(\Omega)$ as the closure of $\mathrm{C}_D^\infty(\Omega)$ in the norm given by (1.2).

For $N \in \mathbb{N}$, let $D_1, \dots, D_N \subset \partial\Omega$ be closed and $\mathbb{D}$ be the N-tuple $\mathbb{D} := (D_1, \dots, D_N)$. Define

$$\mathrm{W}_{\mathbb{D}}^{k,p}(\Omega; \mathbb{C}^N) := \prod_{i=1}^N \mathrm{W}_{D_i}^{k,p}(\Omega)$$

endowed with the usual product norm. If $\mathbb{D} = (\partial\Omega, \dots, \partial\Omega)$, we also write $\mathrm{W}_0^{k,p}(\Omega; \mathbb{C}^N)$.

Remark 1.1.12. The definition for the scalar-valued Sobolev space with partially vanishing trace appears in Ouhabaz [80, Sec. 4.2] and is systematically studied by Haller-Dintelmann, Jonsson, Knees, and Rehberg [47]. In the $\mathbb{C}^N$-valued case with $D_i = D_j$, for $1 \leq i, j \leq N$, these spaces are studied by Brewster, Mitrea, Mitrea, and Mitrea [12].

1.1.3 The distribution function and weak L^p-spaces

Let $\Omega \subset \mathbb{R}^d$ be endowed with the Lebesgue measure. For a measurable function $f : \Omega \to \mathbb{C}$ define the *distribution function* of f, d_f, by

$$d_f : [0, \infty) \to [0, \infty], \quad \alpha \mapsto |\{x \in \Omega : |f(x)| > \alpha\}|.$$

The following proposition lists some elementary properties of the distribution function, see Grafakos [44, Prop. 1.1.3] for the first and third property; the second one directly follows from the definition.

Proposition 1.1.13. *Let f and g be two measurable functions on Ω. Then for all $\alpha, \beta, \gamma > 0$ we have*

(1) $|g| \leq |f|$ *a.e. implies that* $d_g \leq d_f$;

(2) $d_{|f|^\gamma}(\alpha) = d_f(\alpha^{\frac{1}{\gamma}})$;

(3) $d_{f+g}(\alpha+\beta) \leq d_f(\alpha) + d_g(\beta)$.

The distribution function is a useful tool in harmonic analysis, as it encodes information on the L^p-norm of a function.

Proposition 1.1.14 ([44, Prop. 1.1.4]). *For $1 \leq p < \infty$ and $f \in \mathrm{L}^p(\Omega)$, we have*

$$\|f\|^p_{\mathrm{L}^p(\Omega)} = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha.$$

Finally, we define the *weak* $\mathrm{L}^p(\Omega)$*-spaces*.

Definition 1.1.15. For $1 \leq p < \infty$, define the *weak* L^p*-space* $\mathrm{L}^{p,\infty}(\Omega)$ as the set of equivalence classes of measurable functions such that

$$\|f\|_{\mathrm{L}^{p,\infty}(\Omega)} := \sup\{\alpha d_f(\alpha)^{\frac{1}{p}} : \alpha > 0\}$$

is finite.

Note that the $\mathrm{L}^{p,\infty}$-norm fails to fulfill the sharp triangle inequality. Instead it holds

$$\|f+g\|_{\mathrm{L}^{p,\infty}(\Omega)} \leq 2\{\|f\|_{\mathrm{L}^{p,\infty}(\Omega)} + \|g\|_{\mathrm{L}^{p,\infty}(\Omega)}\},$$

so that $\mathrm{L}^{p,\infty}(\Omega)$ turns into a quasi-normed space, see [44, Eq. (1.1.11)]. Moreover, the L^p-spaces are contained in the weak L^p-spaces, as the following proposition shows.

Proposition 1.1.16 ([44, Prop. 1.1.6]). *For all $1 \leq p < \infty$, we have that $\mathrm{L}^p(\Omega) \subset \mathrm{L}^{p,\infty}(\Omega)$ and*

$$\|f\|_{\mathrm{L}^{p,\infty}(\Omega)} \leq \|f\|_{\mathrm{L}^p(\Omega)} \qquad (f \in \mathrm{L}^p(\Omega)).$$

1.1.4 Spaces of solenoidal vector fields

There are solenoidal, i.e., “divergence free”, counterparts of the Lebesgue and Sobolev spaces, which play an important role in the mathematical investigation of the motions of incompressible fluids. To define these spaces, define for an open set $\Omega \subset \mathbb{R}^d$

$$\mathrm{C}_{c,\sigma}^{\infty}(\Omega) := \{\varphi \in \mathrm{C}_c^{\infty}(\Omega;\mathbb{C}^d) : \operatorname{div}(\varphi) = 0\}.$$

For a construction of a non-zero $\mathrm{C}_{c,\sigma}^{\infty}$-function, see the proof of Sohr [90, Lem. II.2.2.1].

Definition 1.1.17. For $1 < p < \infty$, define $\mathrm{L}_{\sigma}^p(\Omega)$ to be the closure of $\mathrm{C}_{c,\sigma}^{\infty}(\Omega)$ in $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ endowed with the L^p-norm. Furthermore, define $\mathrm{W}_{0,\sigma}^{1,p}(\Omega)$ to be the closure of $\mathrm{C}_{c,\sigma}^{\infty}(\Omega)$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{C}^d)$ endowed with the $\mathcal{W}^{1,p}$-norm given by (1.2).

Remark 1.1.18. In the case $p = 2$ it is clear that, by the closedness of $\mathrm{L}_{\sigma}^2(\Omega)$ in $\mathrm{L}^2(\Omega;\mathbb{C}^d)$, there exists an orthogonal complement of $\mathrm{L}_{\sigma}^2(\Omega)$. This can be characterized as

$$\mathrm{L}_{\sigma}^2(\Omega)^{\perp} = \{f \in \mathrm{L}^2(\Omega;\mathbb{C}^d) : f = \nabla g \text{ for some } g \in \mathrm{L}_{\mathrm{loc}}^2(\Omega)\},$$

see [90, Lem. II.2.5.1]. We call the orthogonal projection $\mathbb{P}$ from $\mathrm{L}^2(\Omega;\mathbb{C}^d)$ onto $\mathrm{L}_{\sigma}^2(\Omega)$ the *Helmholtz projection*.

Transferring these facts to the case $p \neq 2$ will in general fail if no further assumptions on either the regularity of Ω or the range of the admissible values of p are made. Fabes, Mendez, and M. Mitrea for example proved the following theorem in [29, Thm. 11.1, Thm. 12.2].

Theorem 1.1.19. *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, there exists a positive number $\varepsilon > 0$, depending on Ω, such that for each p with $3/2 - \varepsilon < p < 3 + \varepsilon$ the operator $\mathbb{P}$ extends to a bounded linear operator from $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ onto $\mathrm{L}_{\sigma}^p(\Omega)$ and the operator $\mathrm{Id} - \mathbb{P}$ extends to a bounded linear operator from $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ onto $\nabla \mathrm{W}^{1,p}(\Omega)$. Hence, in this range*

$$\mathrm{L}^p(\Omega;\mathbb{C}^d) = \nabla \mathrm{W}^{1,p}(\Omega) \oplus \mathrm{L}_{\sigma}^p(\Omega), \tag{1.3}$$

where the direct sum is topological.

On the other hand, for any $p \notin [3/2, 3]$ there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ for which the L^p-Helmholtz decomposition (1.3) *fails.*

Remark 1.1.20. (1) A similar result holds for $d = 2$ and whenever $4/3 - \varepsilon < p < 4 + \varepsilon$, compare D. Mitrea [72, Thm. 4.4].

(2) Note that in [29] the theorem was formulated slightly differently. There, the space $\mathrm{L}^p_\sigma(\Omega)$ was defined as

$$\{u \in \mathrm{L}^p(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0, \langle u, \nu\rangle = 0 \text{ on } \partial\Omega\},$$

where ν denotes the exterior unit normal to $\partial\Omega$ and $\langle \cdot, \cdot \rangle$ the inner product of $\mathbb{C}^d$. However, it is proven in Galdi [34, Thm. III.2.3] that for bounded Lipschitz domains both definitions agree. Furthermore, by [34, Thm. III.4.1], it is proven that

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) = \{u \in \mathrm{W}^{1,p}_0(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\} \tag{1.4}$$

for every $1 < p < \infty$ if Ω is a bounded Lipschitz domain.

In Section 5.1 we will give an alternative and much easier proof of the existence of the L^p-Helmholtz decomposition. On the one hand, the payoff is that in addition to Theorem 1.1.19 we are able to treat the case $d = 2$ and also special Lipschitz domains, which are domains above the graph of a Lipschitz function. On the other hand, the drawback is that our proof only gives the Helmholtz decomposition for p's in the range $\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$, which for $d = 2$ and $d = 3$ agrees with the ranges given above, but becomes more restrictive for $d > 3$. However, this will suffice for our purposes.

1.2 Interpolation

For the introduction of interpolation theory, we rely on the book of Lunardi [67], other standard references are the books of Bergh and Löfström [9] and of Triebel [94].

Let E and F be two real or complex Banach spaces. We say that the couple (E, F) is an *interpolation couple* if both E and F are continuously embedded into a Hausdorff topological space $\mathcal{V}$. For an interpolation couple the sum $E + F := \{e + f : e \in E, f \in F\}$ endowed with the norm

$$\|v\|_{E+F} := \inf\{\|e\|_E + \|f\|_F : e \in E, f \in F, v = e + f\}$$

and the intersection $E \cap F$ endowed with the norm

$$\|e\|_{E\cap F} := \max\{\|e\|_E, \|e\|_F\}$$

are Banach spaces.

1.2.1 The complex interpolation method

In order to introduce the *complex interpolation method*, let (E, F) be an interpolation couple of complex Banach spaces. Define the strip

$$S := \{z = x + \mathrm{i}y \in \mathbb{C} : 0 \leq x \leq 1\}$$

and let $\mathcal{F}(E, F)$ be the space of all functions $g : S \to E + F$ such that

(1) g is holomorphic in the interior of S, and continuous and bounded up to its boundary with respect to the norm of $E + F$;

(2) the function $t \mapsto g(\mathrm{i}t)$ is an element of $\mathrm{C}(\mathbb{R}; E)$ and the function $t \mapsto g(1 + \mathrm{i}t)$ is an element of $\mathrm{C}(\mathbb{R}; F)$, such that the norm

$$\|g\|_{\mathcal{F}(E,F)} := \max\Big\{\sup_{t\in\mathbb{R}} \|g(\mathrm{i}t)\|_E, \sup_{t\in\mathbb{R}} \|g(1 + \mathrm{i}t)\|_F\Big\}$$

is finite.

Now, complex interpolation between E and F is defined as follows.

Definition 1.2.1. For $\theta \in [0, 1]$ define the *complex interpolation space between X and Y with parameter θ* as the set

$$[E, F]_\theta := \{g(\theta) : g \in \mathcal{F}(E, F)\}$$

endowed with the norm

$$\|a\|_{[E,F]_\theta} := \inf\{\|g\|_{\mathcal{F}(E,F)} : g \in \mathcal{F}(E, F) \text{ with } g(\theta) = a\}.$$

This definition yields a Banach space as shown in Lunardi [67, Sec. 2.1]. Before being in the position to present the connections between Sobolev and Bessel potential spaces under complex interpolation, we have to introduce the notion of Banach spaces of class $\mathcal{HT}$.

Definition 1.2.2. A Banach space X is said to be of *class* $\mathcal{HT}$ if the X-valued Hilbert transform

$$(Hf)(t) := \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{t-s} f(s)\, \mathrm{d}s \qquad (f \in \mathcal{S}(\mathbb{R}; X))$$

extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}; X)$ for some $1 < p < \infty$.

Remark 1.2.3. (1) By the boundedness of the classical Hilbert transform, see, e.g., GRAFAKOS [44, Ch. 4], we have that $\mathbb{C}$ is of class $\mathcal{HT}$.

(2) If (Ω, μ) is a σ-finite measure space, $p \in (1, \infty)$, and X of class $\mathcal{HT}$, then $\mathrm{L}^p(\Omega, \mu; X)$ is of class $\mathcal{HT}$ as well, see AMANN [3, Thm. 4.5.2], so that by the first remark all closed subspaces of $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^N)$ are of class $\mathcal{HT}$.

The following theorem is a collection of several results. The first part can be found in HYTÖNEN, VAN NEERVEN, VERAAR, and WEIS [52, Thm. 5.94]. The second part is proven in JERISON and KENIG [53, Prop. 2.4] and the last statement follows from LUNARDI [67, Thm. 2.7].

Theorem 1.2.4. *Let X be a complex Banach space of class $\mathcal{HT}$. Then*

(1) *for all $k \in \mathbb{N}$ and $1 < p < \infty$ the spaces $\mathrm{W}^{k,p}(\mathbb{R}^d; X)$ and $\mathrm{H}^{k,p}(\mathbb{R}^d; X)$ coincide with equivalent norms.*

Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and define for $\theta \in [0, 1]$, $0 \leq s_0, s_1, t_0, t_1 < \infty$ and $1 < p_0, p_1, q_0, q_1 < \infty$

$$s_\theta := s_0(1-\theta) + s_1\theta \qquad \text{and} \qquad \frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$
$$t_\theta := t_0(1-\theta) + t_1\theta \qquad \text{and} \qquad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then, with equivalent norms

(2) $[\mathrm{H}^{s_0,p_0}(\Omega), \mathrm{H}^{s_1,p_1}(\Omega)]_\theta = \mathrm{H}^{s_\theta,p_\theta}(\Omega)$.

If T is a bounded operator from $\mathrm{H}^{s_0,p_0}(\Omega)$ into $\mathrm{H}^{t_0,q_0}(\Omega)$ and from $\mathrm{H}^{s_1,p_1}(\Omega)$ into $\mathrm{H}^{t_1,q_1}(\Omega)$, then T is bounded from $\mathrm{H}^{s_\theta,p_\theta}(\Omega)$ into $\mathrm{H}^{t_\theta,q_\theta}(\Omega)$ and there exists a constant $C > 0$ such that

$$\|T\|_{\mathcal{L}(\mathrm{H}^{s_\theta,p_\theta}(\Omega),\mathrm{H}^{t_\theta,q_\theta}(\Omega))} \leq C \|T\|^{1-\theta}_{\mathcal{L}(\mathrm{H}^{s_0,p_0}(\Omega),\mathrm{H}^{t_0,q_0}(\Omega))} \|T\|^{\theta}_{\mathcal{L}(\mathrm{H}^{s_1,p_1}(\Omega),\mathrm{H}^{t_1,q_1}(\Omega))}.$$

1.2.2 The K- and the trace method

In this subsection, we will give a short glimpse into how real interpolation spaces can be defined via the K-method and present a result which characterizes these spaces as traces at time $t = 0$ of functions in a certain space. This will be of importance in the study of maximal L^q-regularity in Subsection 2.2, by providing a characterization of the set of admissible initial data of certain Banach space valued ordinary differential equations by means of a real interpolation space.

If (X, Y) is an interpolation couple of real or complex Banach spaces, define for every $t > 0$ and $x \in X + Y$ the *K-functional* by

$$K(t, x, X, Y) := \inf_{\substack{x=a+b \\ a\in X, b\in Y}} \{\|a\|_X + t\|b\|_Y\}.$$

The real interpolation spaces are defined as follows.

Definition 1.2.5. Let $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then the *real interpolation space between X and Y with parameter θ and p* is the set

$$(X, Y)_{\theta,p} := \{x \in X + Y : t \mapsto t^{-\theta} K(t, x, X, Y) \in \mathrm{L}^p(0, \infty, \mathrm{d}t/t)\}$$

endowed with the norm

$$\|x\|_{(X,Y)_{\theta,p}} := \|t^{-\theta} K(t, x, X, Y)\|_{\mathrm{L}^p(0,\infty,\mathrm{d}t/t)}.$$

These spaces are Banach spaces; further properties may be found in Lunardi [67, Sec. 1]. An important characterization is the one via the trace method, because this reveals a connection between real interpolation and traces of functions. The Banach spaces connected to the *trace method* are defined as follows.

Definition 1.2.6. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ define $\mathrm{V}(p, \theta, Y, X)$ as the set of all functions $u : (0, \infty) \to X + Y$ with $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}((0, \infty); X + Y)$, and

$$t \mapsto u_\theta(t) := t^\theta u(t) \in \mathrm{L}^p(0, \infty, \mathrm{d}t/t; Y),$$
$$t \mapsto w_\theta(t) := t^\theta u'(t) \in \mathrm{L}^p(0, \infty, \mathrm{d}t/t; X)$$

endowed with the norm

$$\|u\|_{\mathrm{V}(p,\theta,Y,X)} := \|u_\theta\|_{\mathrm{L}^p(0,\infty,\mathrm{d}t/t;Y)} + \|w_\theta\|_{\mathrm{L}^p(0,\infty,\mathrm{d}t/t;X)}.$$

Since $u(t) - u(s) = \int_s^t u'(\sigma)\, \mathrm{d}\sigma$ for $0 < s < t$, one can perform for $1 < p < \infty$ the following estimate

$$\begin{aligned}\|u(t) - u(s)\|_X &\le \int_s^t \|\sigma^{\theta-\frac{1}{p}} u'(\sigma)\|_X \sigma^{\frac{1}{p}-\theta}\, \mathrm{d}\sigma \\ &\le \Big(\int_s^t \|\sigma^\theta u'(\sigma)\|_X^p \,\frac{\mathrm{d}\sigma}{\sigma} \Big)^{\frac{1}{p}} \Big(\int_s^t \sigma^{\frac{p'}{p}-p'\theta}\, \mathrm{d}\sigma \Big)^{\frac{1}{p'}} \\ &\le \|u\|_{\mathrm{V}(p,\theta,Y,X)} \Big(\frac{t^{p'(1-\theta)} - s^{p'(1-\theta)}}{p'(1-\theta)} \Big)^{\frac{1}{p'}},\end{aligned}$$

where p' is the Hölder conjugate exponent of p. Thus, in this case we can infer that the limit $\lim_{t\searrow 0} u(t)$ exists in X. In the case $p = 1$ we have

$$\|u(t) - u(s)\|_X \le \int_s^t \|\sigma^\theta u'(\sigma)\|_X \,\frac{\mathrm{d}\sigma}{\sigma} t^{1-\theta},$$

and in the case $p = \infty$

$$\|u(t) - u(s)\|_X \le \|\sigma^\theta u'(\sigma)\|_{\mathrm{L}^\infty(0,\infty;X)} \int_s^t \sigma^{-\theta}\, \mathrm{d}\sigma,$$

so that the limit also exists in these two cases. Now, we can state the following important characterization, whose proof can be found in [67, Prop. 1.13].

Proposition 1.2.7. *For $0 < \theta < 1$ and $1 \le p \le \infty$, the real interpolation space $(X, Y)_{\theta,p}$ is the set of traces at $t = 0$ of functions in $\mathrm{V}(p, 1-\theta, Y, X)$, and the norm*

$$\|x\|_{\theta,p}^{\mathrm{Tr}} := \inf\{\|u\|_{\mathrm{V}(p,1-\theta,Y,X)} : x = u(0), u \in \mathrm{V}(p, 1-\theta, Y, X)\}$$

is an equivalent norm in $(X, Y)_{\theta,p}$.

1.3 Sobolev spaces on Lipschitz boundaries

We start with the definition of a strong Lipschitz domain to which we will, except for the last chapter of this treatise, simply refer to as a Lipschitz domain. During the whole section, we will use the dash and the double dash notation, i.e., given a vector $x \in \mathbb{R}^d$, x' denotes the vector

$(x_1, \ldots, x_{d-1})$ and x'' denotes the vector $(x_2, \ldots, x_{d-1})$. If $d = 2$, the vector x'' will be void and all terms, in which this vector appears, will be ignored. Furthermore, vectors in $\mathbb{R}^{d-1}$ or in $\mathbb{R}^{d-2}$ will also be tagged with either a dash or a double dash.

Finally, for two sets $A, B \subset \mathbb{R}^d$ we define

$$A \pm B := \{a \pm b : a \in A \text{ and } b \in B\}.$$

Definition 1.3.1. A bounded, open, and connected set $\Omega \subset \mathbb{R}^d$ is called a *bounded strong Lipschitz domain*, or short, *bounded Lipschitz domain*, if there exist $r_0, M > 0$ such that for each point $x \in \partial\Omega$ there exists a Lipschitz continuous function $\eta_x : \mathbb{R}^{d-1} \to \mathbb{R}$ with $\eta_x(0) = 0$ and $\|\nabla \eta_x\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_x : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$

$$\begin{aligned} R_x[\Omega - \{x\}] \cap D(r) &= D_{\eta_x}(r) \\ R_x[\partial\Omega - \{x\}] \cap D(r) &= I_{\eta_x}(r), \end{aligned}$$

where

$$\begin{aligned} D(r) &:= \{(x', x_d) : |x'| < r, |x_d| < 10d(M+1)r\} \\ D_{\eta_x}(r) &:= \{(x', x_d) : |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\} \\ I_{\eta_x}(r) &:= \{(x', x_d) : |x'| < r, \eta_x(x') = x_d\}. \end{aligned}$$

Remark 1.3.2. Vividly speaking, Definition 1.3.1 means that every point $x \in \partial\Omega$ has a neighborhood so that modulo a translation and a rotation the portion of $\partial\Omega$ in this neighborhood is given by the graph of the Lipschitz function η_x. Moreover, this neighborhood is quantified by the cylinders $D(r)$. The portion of the translated and rotated version of Ω that lies inside $D(r)$ is given by $D_{\eta_x}(r)$ and the translated and rotated boundary of Ω that lies inside $D(r)$ is given by $I_{\eta_x}(r)$.

We will call sets of the form $D_\eta(r)$ for a Lipschitz continuous function η with $\|\nabla \eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$ and $r > 0$ *Lipschitz cylinders*. The definition above is exactly the one used by Shen in [89]. Other references for strong Lipschitz domains are Adams and Fournier [1], Grisvard [45], McLean [70], and Nečas [79].

For a bounded Lipschitz domain Ω, $x \in \partial\Omega$, and $0 < r \leq r_0$, let R_x be the rotation according to Definition 1.3.1. Define $U_{x,r} := \{x\} + R_x^{-1} D(r)$. We call the function

$$\Phi_{x,r} : B'(0,r) \to \partial\Omega \cap U_{x,r}, \quad y' \mapsto x + R_x^{-1} \begin{pmatrix} y' \\ \eta_x(y') \end{pmatrix}, \tag{1.5}$$

coordinate function, where $B'(0,r)$ denotes the ball in $\mathbb{R}^{d-1}$ centered in zero and with radius r. If $\Phi_{x,r}$ is a coordinate function, then the set $U_{x,r}$ will in the following always denote the set defined above.

To define the Lipschitz character of a bounded Lipschitz domain Ω, we follow the discussion of Pipher and Verchota in [81, Sec. 5].

Definition 1.3.3. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $x_1, \dots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. Let M be the number from the definition of Lipschitz domains in Definition 1.3.1. We say that a constant $C > 0$ depends on the *Lipschitz character of* Ω if it depends on M and n.

Throughout this section, the following results are of great importance and are used frequently.

Theorem 1.3.4. *Let Ω, $\Xi \subset \mathbb{R}^d$, $d \geq 1$, be open, $f : \Omega \to \mathbb{R}^d$ be Lipschitz continuous, $u \in \mathrm{W}^{1,1}(V)$ for an open neighborhood V of $f(\Omega)$, $v \in \mathrm{L}^1(\Omega)$, and $\varphi : \Xi \to \Omega$ be bi-Lipschitzian. Then*

(1) f is differentiable almost everywhere;

(2) the chain rule holds almost everywhere, i.e., $u \circ f \in \mathrm{W}^{1,1}(\Omega)$ and

$$\partial_j [u \circ f](x) = \langle [\nabla u](x), \partial_j f(x) \rangle$$

holds for almost every $x \in \Omega$. Here and henceforth, for any $l \in \mathbb{N}$ the inner product of $\mathbb{C}^l$ is denoted by $\langle \cdot, \cdot \rangle$;

(3) the change of variables formula holds true, i.e.,

$$\int_\Omega v(x) \, \mathrm{d}x = \int_{\varphi^{-1}(\Omega)} [v \circ \varphi](x) \, |\det(\mathrm{J}_\varphi(x))| \, \mathrm{d}x,$$

where J_φ denotes the Jacobian of φ.

The first part of the theorem above is called Rademacher's theorem and can be found in HEINONEN [50, Thm. 3.1]. The second part is also well known, see, e.g., NEČAS [79, Lem. 3.2] together with [79, Eq. (2.27)] and the third part may be found in HAJŁASZ [48, Thm. 3].

In order to introduce the surface measure, fix finitely many points $x_1, \dots, x_n \in \partial\Omega$ and a radius $0 < r \leq r_0$ such that $(U_{x_i,r})_{i=1}^n$ covers $\partial\Omega$.

Definition 1.3.5. A set $A \subset \partial\Omega$ is *measurable* if for all $1 \leq i \leq n$ the set $\Phi_{x_i,r}^{-1}(A \cap U_{x_i,r})$ is measurable with respect to the $(d-1)$-dimensional Lebesgue measure. The *surface measure* $\sigma(A)$ of this set is then defined as follows. Define

$$A_1 := A \cap U_{x_1,r}, \qquad A_{k+1} := (A \cap U_{x_{k+1},r}) \setminus \bigcup_{i=1}^k A_i \qquad (1 \leq k \leq n-1)$$

and then set

$$\sigma(A) := \sum_{i=1}^n \int_{\Phi_{x_i,r}^{-1}(A_i)} [1 + |\nabla_{y'} \eta_{x_i}(y')|^2]^{\frac{1}{2}} \, \mathrm{d}y'. \tag{1.6}$$

Remark 1.3.6. Proceeding as AMANN and ESCHER did for C^1-domains, see [4, Sec. XII.1], but by taking into account Theorem 1.3.4, one can show that $(\partial\Omega, \sigma)$ turns into a σ-finite measure space and that σ is a complete Radon measure. Furthermore, the same proofs as in [4, Sec. XII.1] show that $u : \partial\Omega \to \mathbb{C}^N$, $N \in \mathbb{N}$, is measurable/integrable if and only if for all $1 \leq i \leq n$ the function $u \circ \Phi_{x_i,r}$ is Lebesgue measurable/integrable with respect to the $(d-1)$-dimensional Lebesgue measure on $B'(0,r)$.

In combination with Theorem 1.3.4, the proof of the following theorem follows the same lines as in [4, Sa. XII.1.11].

Theorem 1.3.7. *If* $(\varphi_i)_{i=1}^n \subset \mathrm{C}_c^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}(\varphi_i) \subset U_{x_i,r}$, $0 \leq \varphi_i \leq 1$, *and* $\sum_{i=1}^n \varphi_i = 1$ *on* $\partial\Omega$ *is a partition of unity subordinate to* $(U_{x_i,r})_{i=1}^n$, *then*

$$\int_{\partial\Omega} u \, \mathrm{d}\sigma = \sum_{i=1}^n \int_{B'(0,r)} [u \circ \Phi_{x_i,r}][\varphi_i \circ \Phi_{x_i,r}][1 + |\nabla_{y'} \eta_{x_i}|^2]^{\frac{1}{2}} \, \mathrm{d}y'$$

for all $u \in \mathrm{L}^1(\partial\Omega, \sigma; \mathbb{C}^N)$.

With the theorem above, we see that the L^p-norm of a function $u \in \mathrm{L}^p(\partial\Omega, \sigma; \mathbb{C}^N)$ has the representation

$$\int_{\partial\Omega} |u|^p \ \mathrm{d}\sigma = \sum_{i=1}^n \int_{B'(0,r)} |u \circ \Phi_{x_i,r}|^p \, [\varphi_i \circ \Phi_{x_i,r}][1 + |\nabla_{y'} \eta_{x_i}|^2]^{\frac{1}{2}} \ \mathrm{d}y'.$$

As the canonical measure on the boundary of a Lipschitz domain is the surface measure, we agree upon the following convention.

Convention 1.3.8. For the rest of this thesis, write $\mathrm{L}^p(\partial\Omega; \mathbb{C}^N)$ for $\mathrm{L}^p(\partial\Omega, \sigma; \mathbb{C}^N)$.

We have seen above, that measurability and integrability reduces to measurability and integrability of the local functions $u \circ \Phi_{x_i,r}$. We proceed in this fashion and define differentiability directly via $u \circ \Phi_{x_i,r}$. This is consistent with the definitions given in Verchota [95, p. 23] and McLean [70, p. 98].

Definition 1.3.9. We say that $u \in \mathrm{L}^1(\partial\Omega; \mathbb{C}^N)$ is *weakly differentiable* if for all $1 \leq i \leq n$ the function $u \circ \Phi_{x_i,r}$ is weakly differentiable on $B'(0, r)$ with weak derivatives in $\mathrm{L}^1(B'(0, r); \mathbb{C}^N)$.

First, we need to verify that weak differentiability is independent of the given set of coordinate functions.

Lemma 1.3.10. *Let $\Phi_{x,t}$ be a coordinate function. If $u \in \mathrm{L}^1(\partial\Omega; \mathbb{C}^N)$ is weakly differentiable, then $u \circ \Phi_{x,t}$ is weakly differentiable on $B'(0, t)$ with weak derivatives in $\mathrm{L}^1(B'(0, t); \mathbb{C}^N)$.*

Proof. Let $(\varphi_i)_{i=1}^n$ be a partition of unity subordinate to $(U_{x_i,r})_{i=1}^n$ and let $\varphi \in \mathrm{C}_c^\infty(B'(0, t))$. Using that Lipschitz functions are differentiable almost everywhere, we obtain

$$[\varphi_i \circ \Phi_{x,t}](y')[\partial_j \varphi](y') = \partial_j[\varphi_i \circ \Phi_{x,t} \cdot \varphi](y') - \partial_j[\varphi_i \circ \Phi_{x,t}](y')\varphi(y')$$

for almost every $y' \in B'(0, t)$. Consequently,

$$\begin{aligned}
&\int_{B'(0,t)} [u \circ \Phi_{x,t}](y')[\partial_j \varphi](y') \ \mathrm{d}y' \\
&\qquad = \sum_{i=1}^n \int_{B'(0,t)} [u \circ \Phi_{x,t}](y')[\varphi_i \circ \Phi_{x,t}](y')[\partial_j \varphi](y') \ \mathrm{d}y' \\
&\qquad = \sum_{i=1}^n \int_{B'(0,t)} [u \circ \Phi_{x,t}](y')\partial_j[\varphi_i \circ \Phi_{x,t} \cdot \varphi](y') \ \mathrm{d}y', \qquad (1.7)
\end{aligned}$$

where the last equality follows by $\nabla \sum_{i=1}^n \varphi_i = 0$, together with

$$\sum_{i=1}^n \partial_j[\varphi_i \circ \Phi_{x,t}](y') = \sum_{i=1}^n \langle [\nabla \varphi_i](\Phi_{x,t}(y')), [\partial_j \Phi_{x,t}](y')\rangle = 0$$

for almost every $y' \in B'(0,t)$.

Continuing in (1.7), we find that $\varphi_i \circ \Phi_{x,t} \cdot \varphi$ is a Lipschitz continuous function with support in $\Phi_{x,t}^{-1}(U_{x_i,r} \cap U_{x,t})$, so that integration in the ith integral takes place in this set. Thus, we can write $u \circ \Phi_{x,t} = [u \circ \Phi_{x_i,r}] \circ [\Phi_{x_i,r}^{-1} \circ \Phi_{x,t}]$. Now, $u \circ \Phi_{x_i,r}$ is in $\mathrm{W}^{1,1}(B'(0,r);\mathbb{C}^N)$ by assumption. Furthermore, $\Phi_{x_i,r}^{-1} \circ \Phi_{x,t} : \Phi_{x,t}^{-1}(U_{x_i,r} \cap U_{x,t}) \to B'(0,r)$ is Lipschitz, one-to-one, and the inverse is also Lipschitz. In this case, Theorem 1.3.4 implies that $u \circ \Phi_{x,t}|_{\Phi_{x,t}^{-1}(U_{x_i,r} \cap U_{x,t})}$ is weakly differentiable. We derive that (1.7) coincides with

$$-\sum_{i=1}^n \int_{B'(0,t)} \partial_j[u \circ \Phi_{x,t}|_{\Phi_{x,t}^{-1}(U_{x_i,r} \cap U_{x,t})}](y')[\varphi_i \circ \Phi_{x,t}](y')\varphi(y')\, \mathrm{d}y'.$$

This concludes the proof. □

Note that $\Phi_{x_i,r}$ is differentiable at all $y' \in B'(0,r)$, where η_{x_i} is differentiable. Moreover, by virtue of (1.5), we deduce that the vectors $\partial_1 \Phi_{x_i,r}(y'), \dots, \partial_{d-1}\Phi_{x_i,r}(y')$ form a basis of a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. This subspace will be called the *tangent space of $\partial\Omega$ at p*, where $p = \Phi_{x_i,r}(y')$. Since

$$\partial_j \Phi_{x_i,r}(y') = R_{x_i}^{-1}[e_j + \partial_j \eta_{x_i}(y') e_d],$$

where e_j denotes the jth standard basis vector, we see that the vector

$$\nu(p) := \frac{1}{[1 + |\nabla_{y'}\eta_{x_i}(y')|^2]^{1/2}} R_{x_i}^{-1} \begin{pmatrix} \nabla_{y'}\eta_{x_i}(y') \\ -1 \end{pmatrix} \tag{1.8}$$

is orthogonal to the tangent space of $\partial\Omega$ at p and points “outside” of Ω. This vector will be called the *outward unit normal to $\partial\Omega$ at p*.

Lemma 1.3.11. *At σ-a.e. $p \in \partial\Omega$ the tangent space at p, and hence the outward unit normal, is independent of the coordinate function. More precisely, given two coordinate functions Φ_{x_1,r_1} and Φ_{x_2,r_2}, with $\Phi_{x_1,r_1}(y') = \Phi_{x_2,r_2}(z') = p$, which are differentiable at y' and z', respectively, we have*

$$\mathrm{span}(\{\partial_j \Phi_{x_1,r_1}(y')\}_{j=1}^{d-1}) = \mathrm{span}(\{\partial_j \Phi_{x_2,r_2}(z')\}_{j=1}^{d-1}).$$

Proof. For simplicity, denote the function Φ_{x_1,r_1} by Φ and U_{x_1,r_1} by U, and similarly Φ_{x_2,r_2} by Ψ and U_{x_2,r_2} by V. Note that if Φ is differentiable at y' and if Ψ is differentiable at z' and $\Phi(y') = \Psi(z')$, we deduce by the chain rule

$$\partial_j \Psi(z') = \mathrm{J}_\Phi(y') \partial_j [\Phi^{-1} \circ \Psi](z'),$$

where J_Φ denotes the Jacobian of Φ, so that

$$\operatorname{span}(\partial_1 \Phi(y'), \ldots, \partial_{d-1} \Phi(y')) \supset \operatorname{span}(\partial_1 \Psi(z'), \ldots, \partial_{d-1} \Psi(z')).$$

Since $\partial_1 \Psi(z'), \ldots, \partial_{d-1} \Psi(z')$ are linearly independent, we have equality. Since Lipschitz continuous functions are differentiable almost everywhere, see Theorem 1.3.4, we conclude the proof by virtue of the definition of the surface measure. □

Suppose that $u : \partial\Omega \to \mathbb{C}$ is weakly differentiable and that $\Phi_{x,r}$ is a coordinate function defined on the set $U_{x,r}$. Let R_x be the rotation according to the definition of $\Phi_{x,r}$ in (1.5). Then, we can define for σ-almost every $p \in U_{x,r} \cap \partial\Omega$ the *tangential gradient of u at p* by

$$\begin{aligned} \nabla_{\tan} u(p) &:= R_x^{-1} \begin{pmatrix} \nabla_{y'} [u \circ \Phi_{x,r}](y') \\ 0 \end{pmatrix} \\ &\quad - \left\langle R_x^{-1} \begin{pmatrix} \nabla_{y'} [u \circ \Phi_{x,r}](y') \\ 0 \end{pmatrix}, \nu(p) \right\rangle \nu(p), \end{aligned} \tag{1.9}$$

where $y' \in B'(0, r)$ is such that $\Phi_{x,r}(y') = p$. Considering this formula, it is not immediately clear which mathematical quantity the tangential gradient describes, except that $\nabla_{\tan} u(p)$ lies in the tangential space of $\partial\Omega$ at p. This follows as it originates from a vector in $\mathbb{R}^d$ whose component in normal direction is subtracted.

In the following proposition, we show that the tangential gradient is well-defined, i.e., that its definition is σ-a.e. independent of the coordinate function. The lemma afterwards, Lemma 1.3.13, will serve to give the right intuition, namely that for smooth functions the tangential gradient coincides with

$$\nabla u(p) - \langle \nabla u(p), \nu(p) \rangle \nu(p).$$

This shows that the choice of the name *tangential gradient* was correct and that the tangential gradient indeed is the projection of the gradient onto the tangential space to $\partial\Omega$.

Proposition 1.3.12. *The tangential gradient is well-defined, i.e., given two coordinate functions Φ_{x_1,r_1} and Φ_{x_2,r_2} with $U_{x_1,r_1} \cap U_{x_2,r_2} \neq \emptyset$, then the tangential gradients defined with respect to Φ_{x_1,r_1} and Φ_{x_2,r_2}, respectively, coincide in $U_{x_1,r_1} \cap U_{x_2,r_2}$ σ-a.e.*

Proof. For simplicity, denote Φ_{x_1,r_1} by Φ, Φ_{x_2,r_2} by Ψ, and $U_{x_1,r_1} \cap U_{x_2,r_2}$ by U.

We agree upon the notation, that $y' \in \Phi^{-1}(U)$ and $z' \in \Psi^{-1}(U)$ are related via $p := \Phi(y') = \Psi(z')$. We already know by Lemma 1.3.11 that the normal vector to $\partial\Omega$ is independent of the coordinate chart, so that $\nu(p)$ can either be written by means of R_{x_1} and η_{x_1} or by means of R_{x_2} and η_{x_2}. Furthermore, recall that

$$\partial_j \Phi(y') = R_{x_1}^{-1}[e_j + \partial_j \eta_{x_1}(y') e_d]$$

if η_{x_1} is differentiable at y'. Also, note that $\Psi^{-1}(p) = (R_{x_2}[p - x_2])'$, where the dash denotes the projection onto the first $d-1$ variables. Consequently, Ψ^{-1} has a smooth extension $\widetilde{\Psi^{-1}}$ to all of $\mathbb{R}^d$ given by the formula above. Now, $\Psi^{-1} \circ \Phi$ and $\widetilde{\Psi^{-1}} \circ \Phi$ coincide on $\Phi^{-1}(U)$, which is an open neighborhood of y', so that we calculate by means of the chain rule

$$\partial_j[\Psi^{-1} \circ \Phi](y') = \partial_j[\widetilde{\Psi^{-1}} \circ \Phi](y') = R'_{x_2} R_{x_1}^{-1}[e_j + \partial_j \eta_{x_1}(y') e_d].$$

With this, another application of the chain rule yields

$$\begin{pmatrix} \nabla_{y'}[u \circ \Phi](y') \\ 0 \end{pmatrix} = \begin{pmatrix} \{R'_{x_2} R_{x_1}^{-1}[e_1 + \partial_1 \eta_{x_1}(y') e_d]\}^T \nabla_{z'}[u \circ \Psi](z') \\ \vdots \\ \{R'_{x_2} R_{x_1}^{-1}[e_{d-1} + \partial_{d-1} \eta_{x_1}(y') e_d]\}^T \nabla_{z'}[u \circ \Psi](z') \\ 0 \end{pmatrix},$$

where the supscript ^{T} denotes the transpose of a matrix and where R'_{x_2} denotes the matrix R_{x_2} without the last row. Use that the inverse of an orthogonal matrix is its transpose and use the notation that a column vector with row vectors as entries denotes a matrix with entries a_{ij} being the jth entry of the row vector in the ith column, to get

$$= \begin{pmatrix} [e_1^T + \partial_1 \eta_{x_1}(y')e_d^T]R_{x_1}(R'_{x_2})^T \\ \vdots \\ [e_{d-1}^T + \partial_{d-1}\eta_{x_1}(y')e_d^T]R_{x_1}(R'_{x_2})^T \\ 0 \end{pmatrix} \nabla_{z'}[u \circ \Psi](z').$$

Adding an additional x_d-component to the vector $\nabla_{z'}[u \circ \Psi](z')$ that is set to zero results in the fact that one can skip the dash at the R_{x_2} matrix, so that

$$= \begin{pmatrix} [e_1^T + \partial_1 \eta_{x_1}(y')e_d^T] \\ \vdots \\ [e_{d-1}^T + \partial_{d-1}\eta_{x_1}(y')e_d^T] \\ 0 \end{pmatrix} R_{x_1} R_{x_2}^T \begin{pmatrix} \nabla_{z'}[u \circ \Psi](z') \\ 0 \end{pmatrix}.$$

Use again that the transpose of an orthogonal matrix is its inverse and rewrite the first matrix by adding and subtracting e_d^T in the last row to obtain

$$= \left[\mathrm{Id} + \begin{pmatrix} 0 & \nabla_{y'}\eta_{x_1}(y') \\ 0 & -1 \end{pmatrix} \right] R_{x_1} R_{x_2}^T \begin{pmatrix} \nabla_{z'}[u \circ \Psi](z') \\ 0 \end{pmatrix}.$$

Multiplying the whole equality by $R_{x_1}^T$ yields an identity for the first term in the definition of the tangential gradient. Note that the term involving the identity matrix is the first term of the tangential gradient of u if it would have been defined with respect to Ψ instead of Φ. It is now the task to prove that the other term cancels with an appropriate term originating in a computation of the second term in (1.9). To compute this term, use the identity above to calculate

$$\begin{aligned}&\left\langle R_{x_1}^{-1}\begin{pmatrix}\nabla_{y'}[u\circ\Phi_{x,r}](y')\\ 0\end{pmatrix},\nu(p)\right\rangle\nu(p)\\ &=\left\langle R_{x_1}^{-1}\left[\operatorname{Id}+\begin{pmatrix}0 & \nabla_{y'}\eta_{x_1}(y')\\ 0 & -1\end{pmatrix}\right]R_{x_1}R_{x_2}^T\begin{pmatrix}\nabla_{z'}[u\circ\Psi](z')\\ 0\end{pmatrix},\nu(p)\right\rangle\nu(p).\end{aligned}$$

In this identity, the term involving the identity matrix is already the second term in the definition of the tangential gradient of u if it would have been defined via Ψ instead of Φ. Thus, we have a closer look at the other term. Using (1.8) and the independence of ν of the given chart, as well that the inverse of an orthogonal matrix is its transpose, we get that this term coincides with

$$\begin{aligned}&\left\langle\begin{pmatrix}0 & \nabla_{y'}\eta_{x_1}(y')\\ 0 & -1\end{pmatrix}R_{x_1}R_{x_2}^T\begin{pmatrix}\nabla_{z'}[u\circ\Psi](z')\\ 0\end{pmatrix},\begin{pmatrix}\nabla_{y'}\eta_{x_1}(y')\\ -1\end{pmatrix}\right\rangle\\ &\qquad\cdot\frac{1}{1+|\nabla_{y'}\eta_{x_1}(y')|^2}R_{x_1}^{-1}\begin{pmatrix}\nabla_{y'}\eta_{x_1}(y')\\ -1\end{pmatrix}.\end{aligned}$$

As the inner product yields a scalar, we can commute the order of multiplication of both lines, additionally, we shift the first matrix on the left-hand side of the inner product to the right-hand side by taking its transpose. This yields

$$=R_{x_1}^{-1}\begin{pmatrix}\nabla_{y'}\eta_{x_1}(y')\\ -1\end{pmatrix}\left\langle R_{x_1}R_{x_2}^T\begin{pmatrix}\nabla_{z'}[u\circ\Psi](z')\\ 0\end{pmatrix},e_d\right\rangle.$$

Write the inner product via $\langle x,y\rangle=y^Tx$ to get

$$=R_{x_1}^{-1}\begin{pmatrix}\nabla_{y'}\eta_{x_1}(y')\\ -1\end{pmatrix}e_d^TR_{x_1}R_{x_2}^T\begin{pmatrix}\nabla_{z'}[u\circ\Psi](z')\\ 0\end{pmatrix}.$$

Finally, note that

$$\begin{pmatrix} \nabla_{y'}\eta_{x_1}(y') \\ -1 \end{pmatrix} e_d^T = \begin{pmatrix} 0 & \nabla_{y'}\eta_{x_1}(y') \\ 0 & -1 \end{pmatrix}.$$

Plugging everything together reveals that

$$\begin{aligned} &R_{x_1}^{-1}\begin{pmatrix} \nabla_{y'}[u\circ\Phi](y') \\ 0 \end{pmatrix} - \left\langle R_{x_1}^{-1}\begin{pmatrix} \nabla_{y'}[u\circ\Phi](y') \\ 0 \end{pmatrix}, \nu(p)\right\rangle \nu(p) \\ &\quad = R_{x_2}^{-1}\begin{pmatrix} \nabla_{y'}[u\circ\Psi](y') \\ 0 \end{pmatrix} - \left\langle R_{x_2}^{-1}\begin{pmatrix} \nabla_{y'}[u\circ\Psi](y') \\ 0 \end{pmatrix}, \nu(p)\right\rangle \nu(p), \end{aligned}$$

which concludes the proof. □

For smooth functions u, we have the following representation of the tangential gradient.

Lemma 1.3.13. *If Ξ is an open neighborhood of $\partial\Omega$ and if $u : \Xi \to \mathbb{C}$ is smooth, then*

$$\nabla_{\tan}u(p) = \nabla u(p) - \langle \nabla u(p), \nu(p)\rangle \nu(p) \qquad (\sigma\text{-a.e. } p \in \partial\Omega).$$

Proof. Since u is smooth in a neighborhood of $\partial\Omega$, we can use the chain rule to deduce

$$\nabla_{y'}[u\circ\Phi_{x,r}](y') = \{[\nabla u](\Phi_{x,r}(y'))^T \mathrm{J}_{\Phi_{x,r}}(y')\}^T = \mathrm{J}_{\Phi_{x,r}}(y')^T[\nabla u](\Phi_{x,r}(y')).$$

Moreover, for almost every y', we can compute $\mathrm{J}_{\Phi_{x,r}}(y')^T$ as

$$\mathrm{J}_{\Phi_{x,r}}(y')^T = \begin{pmatrix} [e_1^T + \partial_1\eta_x(y')e_d^T]R_x \\ \vdots \\ [e_{d-1}^T + \partial_{d-1}\eta_x(y')e_d^T]R_x \end{pmatrix} = \begin{pmatrix} e_1^T + \partial_1\eta_x(y')e_d^T \\ \vdots \\ e_{d-1}^T + \partial_{d-1}\eta_x(y')e_d^T \end{pmatrix} R_x.$$

Plugging this inside the first term of (1.9) yields

$$R_x^{-1}\begin{pmatrix}\nabla_{y'}[u\circ\Phi_{x,r}](y')\\ 0\end{pmatrix} = R_x^{-1}\begin{pmatrix} e_1^T+\partial_1\eta_x(y')e_d^T\\ \vdots \\ e_{d-1}^T+\partial_{d-1}\eta_x(y')e_d^T\\ 0\end{pmatrix} R_x[\nabla u](\Phi_{x,r}(y')).$$

Adding and subtracting e_d^T in the last row shows that this coincides with

$$= R_x^{-1}\left[\mathrm{Id}+\begin{pmatrix}0 & \nabla_{y'}\eta_x(y')\\ 0 & -1\end{pmatrix}\right] R_x[\nabla u](\Phi_{x,r}(y')).$$

The term involving the identity matrix is the desired term $[\nabla u](\Phi_{x,r}(y'))$. We show that the other term cancels during the computation of the second term of (1.9). Here, we find exactly as above

$$\begin{aligned}&\left\langle R_x^{-1}\begin{pmatrix}\nabla_{y'}[u\circ\Phi_{x,r}](y')\\ 0\end{pmatrix}, \nu(p)\right\rangle\nu(p)\\ &= \left\langle R_x^{-1}\left[\mathrm{Id}+\begin{pmatrix}0 & \nabla_{y'}\eta_x(y')\\ 0 & -1\end{pmatrix}\right] R_x[\nabla u](\Phi_{x,r}(y')), \nu(p)\right\rangle\nu(p).\end{aligned}$$

Note that the inverse of an orthogonal matrix is its transpose. This together with (1.8) shows

$$= \langle[\nabla u](\Phi_{x,r}(y')),\nu(p)\rangle\nu(p) + \langle R_x[\nabla u](\Phi_{x,r}(y')), e_d\rangle R_x^{-1}\begin{pmatrix}\nabla_{y'}\eta_x(y')\\ -1\end{pmatrix}.$$

Rewrite the inner product as $\langle x,y\rangle = y^Tx$ and commute the rightmost vector with the inner product to deduce

$$= \langle[\nabla u](\Phi_{x,r}(y')),\nu(p)\rangle\nu(p) + R_x^{-1}\begin{pmatrix}\nabla_{y'}\eta_x(y')\\ -1\end{pmatrix} e_d^T R_x[\nabla u](\Phi_{x,r}(y')).$$

Finally, note that

$$\begin{pmatrix} \nabla_{y'}\eta_x(y') \\ -1 \end{pmatrix} e_d^T = \begin{pmatrix} 0 & \nabla_{y'}\eta_x(y') \\ 0 & -1 \end{pmatrix}.$$

A comparison of the resulting terms yields the representation formula. □

If $u : \partial\Omega \to \mathbb{C}^N$ is weakly differentiable, we define

$$\nabla_{\tan} u := \begin{pmatrix} \nabla_{\tan} u_1 \\ \vdots \\ \nabla_{\tan} u_N \end{pmatrix}.$$

Definition 1.3.14. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $1 \leq p < \infty$. The space $\mathrm{W}^{1,p}(\partial\Omega; \mathbb{C}^N)$ is defined as the set of all $u \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^N)$ which are weakly differentiable and have finite norm

$$\|u\|_{\mathrm{W}^{1,p}(\partial\Omega;\mathbb{C}^N)} := \left[\|u\|^p_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^N)} + \|\nabla_{\tan} u\|^p_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^{(d-1)N})} \right]^{\frac{1}{p}}.$$

Now, that we have defined first-order Sobolev spaces on the boundaries of Lipschitz domains, we continue by introducing notions quantifying the non-tangential behavior of functions defined in Ω. This additional non-tangential behavior often enables us to consider a continuation of this function on $\partial\Omega$ and to deduce some properties of this function on $\partial\Omega$. For this purpose, we introduce the following type of cones. Recall the number M and the cylinder $D(r)$ from the definition of a bounded Lipschitz domain.

Definition 1.3.15. By a *cone*, we mean an open, right circular, nonempty truncated cone (not a double cone).

Assigning a cone $\Gamma(q)$ to each $q \in \partial\Omega$, we call the resulting family $\{\Gamma(q)\}_{q\in\partial\Omega}$ *regular* if there exist finitely many points $x_1, \dots, x_{n_0} \in \partial\Omega$, $\widetilde{r} > 0$, and rotations $\widetilde{R}_{x_1}, \dots, \widetilde{R}_{x_{n_0}}$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5),$$

and such that there exist Lipschitz continuous functions $\widetilde{\eta}_{x_i} : \mathbb{R}^{d-1} \to \mathbb{R}$ such that for all $\widetilde{r} \leq r \leq \nu\widetilde{r}$ with

$$\nu := 1 + \left[1 + [10d(M+1)]^2\right]^{\frac{1}{2}} \tag{1.10}$$

we have

$$\begin{aligned} \widetilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) &= D_{\widetilde{\eta}_{x_i}}(r) \\ \widetilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) &= I_{\widetilde{\eta}_{x_i}}(r). \end{aligned}$$

Moreover, for each i there are three cones α_i, β_i, and γ_i, each with vertex at the origin and axis along the x_d-axis, such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i \tag{1.11}$$

and such that for all $q \in [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5)] \cap \partial\Omega$, we have

$$\begin{gathered} \widetilde{R}_{x_i}^{-1}\alpha_i + \{q\} \subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} \subset \widetilde{R}_{x_i}^{-1}\beta_i + \{q\}, \\ \widetilde{R}_{x_i}^{-1}\gamma_i + \{q\} \subset [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\widetilde{r})] \cap \Omega. \end{gathered} \tag{1.12}$$

A cone $\Gamma(q)$ of a regular family of cones will be called a *regular cone*.

Remark 1.3.16. (1) In general, the rotations R_{x_1} and Lipschitz functions η_{x_i} from Definition 1.3.1 and $\widetilde{R}_{x_i}$ and $\widetilde{\eta}_{x_i}$ differ. This originates from the following two facts.

First of all, the axis of revolution of α_i is assumed to be along the x_d-axis, so that by (1.12) a regular cone $\Gamma(q)$ contains the axis of revolution of the rotated cylinder $\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\widetilde{r})$. Secondly, $\Gamma(q)$ is independent of i, so that the previous statement must be valid for all $1 \leq i \leq n_0$ such that $q \in [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5)] \cap \partial\Omega$. If we would have taken the rotations R_{x_i} instead of $\widetilde{R}_{x_i}$, it could happen that for two nearby points x_i and x_j, the rotations R_{x_i} and R_{x_j} differ significantly, so that it may not be achievable that $\Gamma(q)$ contains the axes of revolution of the cylinders $\{x_i\} + R_{x_i}^{-1} D(\widetilde{r})$ and $\{x_j\} + R_{x_j}^{-1} D(\widetilde{r})$. This makes it inevitable to introduce the rotations $\widetilde{R}_{x_i}$ and $\widetilde{R}_{x_j}$, which ensure that the rotations of two nearby points do not differ too much.

(2) The number n_0 can be much larger that the number n from the definition of the Lipschitz character, Definition 1.3.3.

(3) For the existence of such a regular family of cones, see the proof of [95, Thm. A.1] in Verchota's thesis. An inspection of this proof reveals that this particular regular family of cones, constructed by Verchota, has some further properties. Namely, the heights h of the cones $\Gamma(q)$ are independent of q and are equal to $C_{d,M}\widetilde{r}$, where $C_{d,M}$ is a constant that depends only on d and M. Moreover, if ϑ_{α_i}, ϑ_{β_i}, and ϑ_{γ_i} denote the opening angles of the cones α_i, β_i, and γ_i, then these angles depend only on the Lipschitz constant M. Furthermore, the Lipschitz constants of the functions $\widetilde{\eta}_{x_i}$ are all bounded by a constant depending only on M. Finally, the thickness of the boundary strip

$$\Omega \cap \Big[\bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5) \Big]$$

is comparable to d, M, and $\widetilde{r}$, i.e.,

$$\operatorname{dist}\Big(\partial\Omega, \Omega \setminus \Big[\bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5) \Big]\Big) \geq C,$$

where $C > 0$ depends only on d, M, and $\widetilde{r}$.

(4) Note that the regular family of cones lies inside Ω. It was also proven in [95, Thm. A.1] that for every bounded Lipschitz domain, there exists a regular family of cones $\{\Gamma(q)\}_{q\in\partial\Omega}$, which lies outside of Ω. This family satisfies the same conditions as above, with the modification that the cones α_i, β_i, and γ_i are given as the reflections of the cones above at the $\{x_d = 0\}$-hyperplane, and that the second line in (1.12) has to be replaced by

$$\widetilde{R}_{x_i}^{-1}\gamma_i + \{q\} \subset [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\widetilde{r})] \setminus \Omega.$$

Now, we can define the notion of the non-tangential maximal function and of non-tangential convergence.

Definition 1.3.17. Given a regular family of cones $\{\Gamma(q)\}_{q\in\partial\Omega}$, which lies either inside or outside Ω and a function $u : \Omega \to \mathbb{C}^N$, $N \in \mathbb{N}$, the *non-tangential maximal function* is defined as

$$(u)^*(q) := \sup_{x\in\Gamma(q)} |u(x)| \qquad (q \in \partial\Omega).$$

For a point $q \in \partial\Omega$ and a vector $c \in \mathbb{C}^N$ we say that *u converges in q non-tangentially to c* if for any regular cone $\Gamma(q)$, the limit

$$\lim_{\substack{x\to q \\ x\in\Gamma(q)}} u(x) = c$$

exists.

Remark 1.3.18. (1) The non-tangential maximal function depends on the respective regular family of cones. If in a given statement we simply write $(u)^* \in \mathrm{L}^p(\partial\Omega)$, we mean that there exists a regular family of cones $\{\Gamma(q)\}_{q\in\partial\Omega}$, such that the non-tangential maximal function of u defined with respect to $\{\Gamma(q)\}_{q\in\partial\Omega}$ lies in $\mathrm{L}^p(\partial\Omega)$. If in a given statement the non-tangential maximal functions of two or more functions occur, then the respective regular families of cones are allowed to differ.

(2) Non-tangential limits from inside and outside Ω do in general not coincide. To differ the two limits in a given situation, the limit will be denoted by the subscript $+$ if the limit is taken from inside Ω and by the subscript $-$ if the limit is taken from outside Ω.

With exception of Chapter 4, we will only consider regular families of cones, non-tangential maximal functions, and non-tangential limits from inside Ω, so that for the rest of this chapter, all these notions are taken from inside Ω.

If a function u on Ω has a suitable non-tangential behavior, one can often transfer some properties to a continuation of u onto the boundary. To do so, the following approximation, which is due to Verchota [95, Thm. A.1], of Ω by smooth domains is an important tool.

Proposition 1.3.19. *Let Ω be a bounded Lipschitz domain with associated numbers r_0 and M from Definition 1.3.1 and recall the number ν from* (1.10)*. Then the following holds.*

(i) *There is a sequence of* C^∞*-domains,* $\Omega_k \subset \Omega$*, and homeomorphisms,* $\Lambda_k : \partial\Omega \to \partial\Omega_k$*, such that*

$$\sup_{q\in\partial\Omega} |q - \Lambda_k(q)| \to 0 \quad \textit{as} \quad k \to \infty.$$

(ii) *There exist* $x_1, \dots, x_{n_0} \in \partial\Omega$*,* $\widetilde{r} > 0$*, and rotations* $\widetilde{R}_{x_1}, \dots, \widetilde{R}_{x_{n_0}}$ *such that*

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5).$$

Moreover, for all $1 \le i \le n_0$ *there are Lipschitz continuous functions* $\widetilde{\eta}_{x_i} : \mathbb{R}^{d-1} \to \mathbb{R}$ *with* $\widetilde{\eta}_{x_i}(0) = 0$ *such that for all* $k \in \mathbb{N}$

$$\begin{aligned}
\widetilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(\nu\widetilde{r}) &= D_{\widetilde{\eta}_{x_i}}(\nu\widetilde{r}) \\
\widetilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(\nu\widetilde{r}) &= I_{\widetilde{\eta}_{x_i}}(\nu\widetilde{r}) \\
\widetilde{R}_{x_i}[\Omega_k - \{x_i\}] \cap D(\nu\widetilde{r}) &= D_{\psi_k^i}(\nu\widetilde{r}) \\
\widetilde{R}_{x_i}[\partial\Omega_k - \{x_i\}] \cap D(\nu\widetilde{r}) &= I_{\psi_k^i}(\nu\widetilde{r}),
\end{aligned}$$

where ψ_k^i *are* C^∞*-functions on* $\mathbb{R}^{d-1}$ *with* $\psi_k^i \to \eta_{x_i}$ *uniformly as* $k \to \infty$*,* $\|\nabla_{y'}\psi_k^i\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \le \|\nabla_{y'}\eta_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$ *for all* $k \in \mathbb{N}$*, and* $\nabla_{y'}\psi_k^i \to \nabla_{y'}\eta_{x_i}$ *pointwise a.e. as* $k \to \infty$*.*

Remark 1.3.20. An analysis of VERCHOTA'S proof reveals that the quantities $x_1, \dots, x_{n_0}$, $\widetilde{r}$, $\widetilde{R}_{x_1}, \dots, \widetilde{R}_{x_{n_0}}$, and $\widetilde{\eta}_{x_1}, \dots, \widetilde{\eta}_{x_{n_0}}$ in Proposition 1.3.19 coincide with the analogous quantities of the regular family of cones, whose construction was mentioned in Remark 1.3.16 (3).

The following lemma is the counterpart of Lemma 1.3.13 in the case where $u : \Omega \to \mathbb{C}$ is only smooth inside Ω but has additional non-tangential behavior.

Lemma 1.3.21. *Fix* $1 \le p < \infty$ *and* $\{\Gamma(q)\}_{q\in\partial\Omega}$ *being the regular family of cones from Remark 1.3.16 (3). Define the non-tangential maximal function with respect to this regular family of cones. Let* $u : \Omega \to \mathbb{C}$ *be a smooth function with* $(u)^*, (\nabla u)^* \in \mathrm{L}^p(\partial\Omega)$*, such that* u *and* ∇u *converge* *non-tangentially to functions* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *for almost every* $q \in \partial\Omega$. *Then* $\boldsymbol{u} \in \mathrm{W}^{1,p}(\partial\Omega)$,

$$\nabla_{\tan}\boldsymbol{u}(q) = \boldsymbol{v}(q) - \langle \boldsymbol{v}(q), \nu(q)\rangle \nu(q) \qquad (\sigma\text{-a.e. } q \in \partial\Omega).$$

and

$$\|\boldsymbol{u}\|_{\mathrm{L}^p(\partial\Omega)} \leq \|(u)^*\|_{\mathrm{L}^p(\partial\Omega)}, \quad \|\nabla_{\tan}\boldsymbol{u}\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)} \leq 2\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)}.$$

Proof. Let Ω_k be a sequence of smooth domains, approximating Ω in the sense of Proposition 1.3.19, and let ν, $x_1, \dots, x_{n_0}$, $\widetilde{r} > 0$, $\widetilde{R}_{x_1}, \dots, \widetilde{R}_{x_{n_0}}$, and $\widetilde{\eta}_{x_1}, \dots, \widetilde{\eta}_{x_{n_0}}$ be the quantities from Proposition 1.3.19. Then there are smooth functions ψ_k^i such that for all $1 \leq i \leq n_0$

$$\begin{aligned} \widetilde{R}_{x_i}[\Omega_k - \{x_i\}] \cap D(\nu\widetilde{r}) &= D_{\psi_k^i}(\nu\widetilde{r}) \\ \widetilde{R}_{x_i}[\partial\Omega_k - \{x_i\}] \cap D(\nu\widetilde{r}) &= I_{\psi_k^i}(\nu\widetilde{r}). \end{aligned}$$

Define analogously to (1.5) the coordinate functions

$$\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y') := x_i + \widetilde{R}_{x_i}^{-1}\begin{pmatrix} y' \\ \widetilde{\eta}_{x_i}(y') \end{pmatrix} \quad \text{and} \quad \Psi_k^i(y') := x_i + \widetilde{R}_{x_i}^{-1}\begin{pmatrix} y' \\ \psi_k^i(y') \end{pmatrix}.$$

Since Ω_k is inside Ω, we have that $\psi_k^i(y')$ is greater than $\widetilde{\eta}_{x_i}(y')$. Combining this with the fact, that any regular cone contains the axis parallel to the axis of revolution of the local coordinate cylinder, see (1.12), ensures that for every $y' \in B'(0, 4\widetilde{r}/5)$ the point $\Psi_k^i(y')$ lies inside the cone $\Gamma(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y'))$. By virtue of the non-tangential convergence, this results in the fact that for almost every y', we have convergence of $u(\Psi_k^i(y'))$ to $\mathbf{u}(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y'))$ and of $[\nabla u](\Psi_k^i(y'))$ to $\mathbf{v}(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y'))$. Furthermore, we have

$$\begin{aligned} |u(\Psi_k^i(y'))| &\leq (u)^*(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')), \\ |[\nabla u](\Psi_k^i(y'))| &\leq (\nabla u)^*(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')). \end{aligned} \tag{1.13}$$

Since $\partial\Omega_k$ lies inside Ω, and since u and Ψ_k^i are smooth, we conclude that for $\varphi \in \mathrm{C}_c^\infty(B'(0,4\widetilde{r}/5))$ the formula

$$\begin{aligned}\int_{B'(0,4\widetilde{r}/5)} & u(\Psi_k^i(y'))\partial_j\varphi(y')\,\mathrm{d}y' \\ &= -\int_{B'(0,4\widetilde{r}/5)} \langle [\nabla u](\Psi_k^i(y')), \partial_j \Psi_k^i(y')\rangle \varphi(y')\,\mathrm{d}y' \\ &= -\int_{B'(0,4\widetilde{r}/5)} \Big\{\langle \widetilde{R}_{x_i}[\nabla u](\Psi_k^i(y')), \partial_j\psi_k^i(y')e_d\rangle \\ &\qquad + \langle \widetilde{R}_{x_i}[\nabla u](\Psi_k^i(y')), e_j\rangle\Big\}\varphi(y')\,\mathrm{d}y'\end{aligned}$$

holds for $1 \leq j \leq d-1$. In the last line we used that the inverse of an orthogonal matrix is its transpose. Next, note that $\|\nabla_{y'}\psi_k^i\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$ and that $\nabla_{y'}\psi_k^i \to \nabla_{y'}\eta_{x_i}$ pointwise almost everywhere. Combining this with (1.13) and the integrability of the non-tangential maximal functions shows that the dominated convergence theorem is applicable, which yields

$$\begin{aligned}\int_{B'(0,4\widetilde{r}/5)} & \mathbf{u}(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y'))\partial_j\varphi(y')\,\mathrm{d}y' \\ &= -\int_{B'(0,4\widetilde{r}/5)} \Big\{\langle \widetilde{R}_{x_i}\mathbf{v}\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')), \partial_j\eta_{x_i}(y')e_d\rangle \\ &\qquad + \langle \widetilde{R}_{x_i}\mathbf{v}(\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')), e_j\rangle\Big\}\varphi(y')\,\mathrm{d}y'.\end{aligned}$$

Next, note that the L^p-integrability of $\mathbf{u}$ and $\mathbf{v}$ follows by $|\mathbf{u}(q)| \leq (u)^*(q)$ and $|\mathbf{v}(q)| \leq (\nabla u)^*(q)$ for almost every $q \in \partial\Omega$ (this is a consequence of the non-tangential convergence and the definition of the non-tangential maximal function). We deduce that $\mathbf{u} \in \mathrm{W}^{1,p}(\partial\Omega)$. To prove the representation of the tangential gradient, note that u is smooth in a neighborhood of $\partial\Omega_k$. Consequently, by Lemma 1.3.13, the tangential gradient of u on $\partial\Omega_k$ has the representation

$$\begin{aligned}\widetilde{R}_{x_i}^{-1}\begin{pmatrix}\nabla_{y'}[u\circ\Psi_k^i](y') \\ 0\end{pmatrix} - \Big\langle \widetilde{R}_{x_i}^{-1}\begin{pmatrix}\nabla_{y'}[u\circ\Psi_k^i](y') \\ 0\end{pmatrix}, \nu_k(\Psi_k^i(y'))\Big\rangle \nu_k(\Psi_k^i(y')) \\ = \nabla u(\Psi_k^i(y')) - \langle \nabla u(\Psi_k^i(y')), \nu_k(\Psi_k^i(y'))\rangle \nu_k(\Psi_k^i(y')),\end{aligned}$$

where ν_k denotes the outward unit normal to $\partial\Omega_k$. By the non-tangential convergence of u and its gradient as well as the convergence of $\Psi_k^i(y')$

to $\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')$, the convergence a.e. of $\nabla_{y'}\Psi_k^i(y')$ to $\nabla_{y'}\widetilde{\Phi}_{x_i,\nu\widetilde{r}}(y')$, and the convergence a.e. of the outward unit normals to ν, cf. (1.8), we conclude the proof by means of Proposition 1.3.12. $\square$

If $\mathfrak{g}$ is a representative of $g \in \mathrm{L}^p(B(0,1))$ an application of polar coordinates yields

$$\int_{B(0,1)} |\mathfrak{g}|^p \, \mathrm{d}x = \int_0^1 \int_{\partial B(0,r)} |\mathfrak{g}|^p \, \mathrm{d}\sigma \, \mathrm{d}r < \infty. \tag{1.14}$$

Thus, for almost every $r \in (0,1)$ the function $\mathfrak{g}|_{\partial B(0,r)}$ is $\mathrm{L}^p(\partial B(0,r))$-integrable. We see that even though one fails to assign a trace to g on a *prescribed* subsphere of $B(0,1)$ (because of the lack of smoothness) one can at least take the trace of every representative in such a way, such that they inherit the L^p-integrability on a whole bunch of seemingly *coincidentally distributed* subspheres of $B(0,1)$. In given problems, this can be important if one can exploit the information $\mathfrak{g}|_{\partial B(0,r)} \in \mathrm{L}^p(\partial B(0,r))$ in order to deduce further estimates. By (1.14), estimates on almost every subsphere result in an estimate on g. Examples for such a reasoning will follow in Chapter 5.

The following theorem gives the framework for this integration argument on more complicated sets. For a proof consult Federer [31, Thm. 3.2.12].

Theorem 1.3.22 (Co-area formula)**.** *If $f : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, is Lipschitzian and $\mathfrak{g}$ the representative of a function $g \in \mathrm{L}^1(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \mathfrak{g}(x) \Big[\sum_{i=1}^d |\partial_i f(x)|^2 \Big]^{\frac{1}{2}} \, \mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(y)} \mathfrak{g}(x) \, \mathrm{d}m_{d-1}(x) \, \mathrm{d}y,$$

where m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$.

Our aim is to use this theorem in order to deduce a technical proposition which resembles the theorem on absolute continuity of $\mathrm{W}^{1,p}(\Omega)$-functions on almost all straight lines parallel to the coordinate axis in Ω, see, e.g., Maz'ya [69, Thm. 1.1.3.1]. Here, we will need that the Lipschitz cylinder $D_\eta(r)$ itself is a Lipschitz domain for every Lipschitz continuous function η and $r > 0$. We will prove this in the following three lemmas. Recall that we employ the one dash and the double dash notation to indicate $x' := (x_1, \dots, x_{d-1})$ and $x'' := (x_2, \dots, x_{d-1})$.

Lemma 1.3.23. *Let* $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ *be a Lipschitz continuous function and* R_α *be a rotation in the* x_1-x_d*-plane about the angle* α *with* $\tan(|\alpha|) < 1/\|\nabla_{x'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$*, then*

$$\{R_\alpha(x', \eta(x')) : x' \in \mathbb{R}^{d-1}\}$$

is the graph of a Lipschitz function, whose Lipschitz constant solely depends on α *and* $\|\nabla_{x'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$.

Proof. Applying the rotation to the elements $(x', \eta(x'))$ in the graph, we deduce

$$R_\alpha(x', \eta(x')) = (x_1 \cos(\alpha) - \eta(x')\sin(\alpha), x'', x_1 \sin(\alpha) + \eta(x')\cos(\alpha)).$$

Assuming the rotated graph is not a graph, we find $x', y' \in \mathbb{R}^{d-1}$ with $x'' = y''$,

$$x_1 \cos(\alpha) - \eta(x')\sin(\alpha) = y_1 \cos(\alpha) - \eta(y')\sin(\alpha), \tag{1.15}$$

and

$$x_1 \sin(\alpha) + \eta(x')\cos(\alpha) \neq y_1 \sin(\alpha) + \eta(y')\cos(\alpha). \tag{1.16}$$

From (1.16), we derive that $x_1 \neq y_1$ by the following reasoning. If $x_1 = y_1$ we find together with $x'' = y''$ that $x' = y'$. But this contradicts (1.16) so that x_1 and y_1 have to be different. From (1.15), we infer that

$$|\eta(x') - \eta(y')| = \frac{|x' - y'|}{\tan(|\alpha|)} > \|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \left|x' - y'\right|,$$

which is a contradiction to the Lipschitz continuity of η. Consequently, the rotated graph is still a graph.

To prove that this graph is the graph of a Lipschitz function, let $x', y' \in \mathbb{R}^{d-1}$ and estimate by means of the triangle inequality, the Lipschitz continuity of η, and $|x_1 - y_1| \leq |x' - y'|$

$$\begin{aligned}&\Big|x_1 \sin(\alpha) + \eta(x')\cos(\alpha) - (y_1 \sin(\alpha) + \eta(y')\cos(\alpha))\Big| \\ &\qquad\qquad \leq \Big[\, |\sin(\alpha)| + |\cos(\alpha)| \, \|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\Big] \left|x' - y'\right|.\end{aligned}$$

Now, there appears $|x' - y'|$ on the right-hand side of the last inequality. However, since the "new" x_1 variable of the rotated coordinate system is $x_1 \cos(\alpha) - \eta(x') \sin(\alpha)$, the Lipschitz estimate follows if we can find a constant $C > 0$ such that

$$\begin{aligned}|x_1 - y_1| \leq C\Big\{&\Big|x_1 \cos(\alpha) - \eta(x') \sin(\alpha) - (y_1 \cos(\alpha) - \eta(y') \sin(\alpha))\Big| \\ &+ |x'' - y''|\Big\}.\end{aligned}$$

This follows by

$$\begin{aligned}|x_1 - y_1| \leq &\frac{|x_1 \cos(\alpha) - \eta(x') \sin(\alpha) - (y_1 \cos(\alpha) - \eta(y') \sin(\alpha))|}{|\cos(\alpha)|} \\ &+ \tan(|\alpha|)\,|\eta(x') - \eta(y')|\end{aligned}$$

and by observing that the last term is estimated by

$$\begin{aligned}\tan(|\alpha|)\,|\eta(x') - \eta(y')| &\leq \tan(|\alpha|)\|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\left[\sum_{i=1}^{d-1} |x_i - y_i|^2\right]^{\frac{1}{2}} \\ &\leq \tan(|\alpha|)\|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\sum_{i=1}^{d-1} |x_i - y_i|\,.\end{aligned}$$

By assumption $\tan(|\alpha|)\|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} < 1$, so that we can absorb the term $\tan(|\alpha|)\|\nabla_{z'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\,|x_1 - y_1|$ to the left-hand side and thereby obtain the estimate. □

Lemma 1.3.24. *Let $A_1, A_2 \subset \overline{B'(0,r)}$ be two nonempty, closed sets with disjoint interiors and $A_1 \cup A_2 = \overline{B'(0,r)}$. Let $\eta_1 : A_1 \to \mathbb{R}$ and $\eta_2 : A_2 \to \mathbb{R}$ be two Lipschitz continuous functions with $\eta_1 = \eta_2$ on $A_1 \cap A_2$ and Lipschitz constants L_1, L_2. Then*

$$\eta : \overline{B'(0,r)} \to \mathbb{R}, \quad x' \mapsto \begin{cases} \eta_1(x'), & \textit{if } x' \in A_1 \\ \eta_2(x'), & \textit{if } x' \in A_2 \end{cases}$$

is Lipschitz continuous with Lipschitz constant bounded by $\max\{L_1, L_2\}$.

Proof. If x' and y' are in either of the sets A_1 or A_2, the Lipschitz estimate is clear. Let $x' \in A_1$ and $y' \in A_2$. Then there exists $t_0 \in [0,1]$ such that the convex combination $z' := t_0 x' + (1-t_0)y'$ is contained in $A_1 \cap A_2$. This follows for example by applying the intermediate value theorem to the continuous function

$$[0,1] \ni t \mapsto \operatorname{dist}(tx' + (1-t)y', A_1) - \operatorname{dist}(tx' + (1-t)y', A_2).$$

Now, estimate

$$\begin{aligned} |\eta(x') - \eta(y')| &\leq |\eta_1(x') - \eta_1(z')| + |\eta_2(z') - \eta_2(y')| \\ &\leq \max\{L_1, L_2\}(|x'-z'| + |z'-y'|) \\ &= \max\{L_1, L_2\}\, |x'-y'|\, . \end{aligned} \qquad \square$$

In the following lemma, we will prove that a Lipschitz cylinder $D_\eta(r)$ itself is a Lipschitz domain. For this, we assume that $\|\nabla_{x'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$ for a fixed number $M > 0$. By virtue of Definition 1.3.1, we have to cover the boundary of $D_\eta(r)$ by cylinders $U_{x,\widetilde{r}}$, in which $U_{x,\widetilde{r}} \cap \partial D_\eta(r)$ can be described as the graph of a Lipschitz continuous function. In order to avoid multiple usage of the same notation, we will denote the Lipschitz constant of these functions by $\widetilde{M}$. As the height of the cylinders $U_{x,\widetilde{r}}$ depends on $\widetilde{M}$, see Definition 1.3.1, we will tag the unrotated and untranslated cylinder by a tilde as well, i.e., we define

$$\widetilde{D}(\widetilde{r}) := \{(y', y_d) : |y'| < \widetilde{r}, |y_d| < 10d(\widetilde{M}+1)\widetilde{r}\} \qquad (\widetilde{r} > 0).$$

To be consistent with this notation, we will write $\widetilde{\eta}_x$ for the Lipschitz function, which describes the boundary of $D_\eta(r)$ in a neighborhood of $x \in \partial D_\eta(r)$ and $\widetilde{R}_x$ for the corresponding rotation. Moreover, we will write $\widetilde{r}_0$ for the number r_0 in Definition 1.3.1.

Finally, we fix the notation that T denotes the top of the cylinder $D_\eta(r)$ and that B denotes its cylinder barrel, i.e.,

$$\begin{aligned} T &:= \{(y', 10d(M+1)r) : |y'| < r\} \\ B &:= \partial D_\eta(r) \setminus \{I_\eta(r) \cup T\}. \end{aligned}$$

Lemma 1.3.25. *Let $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ be Lipschitz continuous with $\eta(0) = 0$ and $\|\nabla_{x'}\eta\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and let $r \geq r' > 0$. Then the Lipschitz cylinder*

$D_\eta(r)$ is a bounded Lipschitz domain with Lipschitz constant $\widetilde{M}$ depending only on M and d, and with $\widetilde{r}_0$ depending only on M, d, and r'.

Let $x = (y', \eta(y'))$ for some $|y'| = r$ and let $\widetilde{R}_x$ be the rotation, such that $\widetilde{D}(\widetilde{r}_0) \cap \widetilde{R}_x[\partial D_\eta(r) - \{x\}]$ is represented by the Lipschitz function $\widetilde{\eta}_x$.

Then, if $d \geq 3$, there exists a Lipschitz continuous function ϑ defined on $\{y'' \in \mathbb{R}^{d-2} : |y''|^2 \leq 15r^2/16\}$ such that $z \in \widetilde{R}_x[B - \{x\}]$ and $|z'| < \widetilde{r}_0$ if and only if

$$z' \in \{w' : w_1 \leq \vartheta(w''), |w'| < \widetilde{r}_0\}.$$

If $d = 2$ and $x_1 = -r$, then $z \in \widetilde{R}_x[B - \{x\}]$ and $|z'| < \widetilde{r}_0$ if and only if

$$-\widetilde{r}_0 < z_1 \leq 0.$$

If $x_1 = r$, then $z \in \widetilde{R}_x[B - \{x\}]$ and $|z'| < \widetilde{r}_0$ if and only if

$$0 \leq z_1 < \widetilde{r}_0.$$

Remark 1.3.26. The statement of the second part of Lemma 1.3.25 is that in the local representation of $\partial D_\eta(r)$ around a point in the corner at the bottom, the preimages of the described portions of the cylinder barrel B and of $I_\eta(r)$ under the coordinate function are separated by the graph of a Lipschitz function, see Figure 1.

Proof of Lemma 1.3.25. We have to show the existence of $\widetilde{r}_0, \widetilde{M} > 0$ such that around every $x \in \partial D_\eta(r)$, the boundary of $D_\eta(r)$ can be represented as the graph of a Lipschitz function $\widetilde{\eta}_x$ in the sense of Definition 1.3.1. For the whole proof, we assume that x is in the x_1-x_d-plane with $x_1 \leq 0$. In two dimensions, this is achieved by a possible reflection of the whole domain at the x_2-axis. Note that a reflection is not a rotation, so that after performing the proof one must not forget to reflect all derived quantities again. In three or more dimensions, this is achieved by proceeding as follows. Write $x = (x', x_d)$ and if $x' = 0$, do nothing. Else, perform a rotation $R^{(1)}$ in the x_{d-2}-x_{d-1}-plane, which maps the components (x_{d-2}, x_{d-1}) onto a vector of the form $(y, 0)$. Proceed by doing the same with the resulting vector $R^{(1)}x$, but with a rotation in the x_{d-3}-x_{d-2}-plane and continue until the rotation in the x_1-x_2-plane is performed. The desired rotation is then given by $R^{(d-2)} \cdot \ldots \cdot R^{(1)}$.

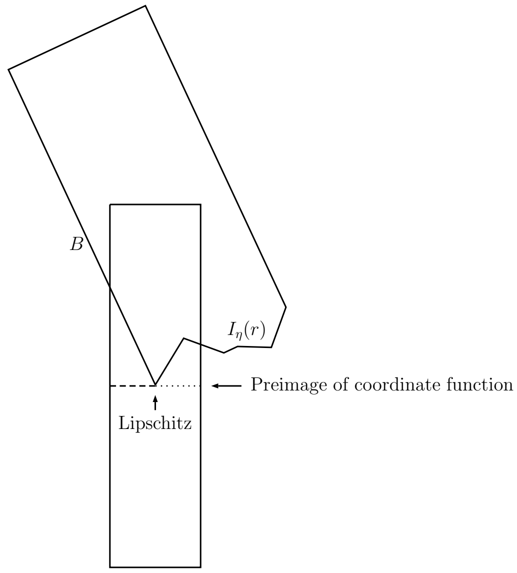

Figure 1: The dashed line indicates the preimage of the cylinder barrel under the coordinate function, the dotted line the preimage of the rotated Lipschitz graph under the coordinate function. The interface separating those preimages is the graph of a Lipschitz function.

The first restriction imposed on the number $\widetilde{r}_0$ will be to assume that $\widetilde{r}_0 < r$.

Case 1: $x \in T$ **with** $|x'| < r - \widetilde{r}_0$**.**

First, note that since $\eta(0) = 0$ the Lipschitz function η is trapped between $-Mr$ and Mr.

Thus, if the bottom of $\{x\} + \widetilde{D}(\widetilde{r}_0)$ stays above the set $\{y_d = Mr\}$, the set $\{x\} + \widetilde{D}(\widetilde{r}_0)$ is disected in only two pieces, one inside and one outside of $D_\eta(r)$. This happens if and only if

$$Mr \leq 10d(M+1)r - 10d(\widetilde{M}+1)\widetilde{r}_0,$$

which is equivalent to

$$10d(\widetilde{M}+1)\widetilde{r}_0 \leq (10d(M+1) - M)r. \tag{1.17}$$

Consequently, if $\widetilde{r}_0$ and $\widetilde{M}$ are chosen such that this inequality is satisfied, we can perform a rotation in the x_1-x_d-plane by 180 degrees and, modulo

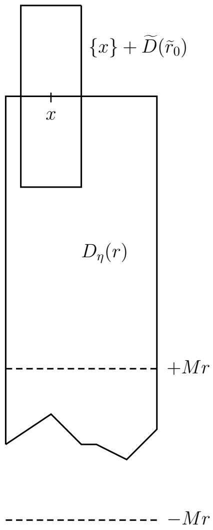

Figure 2: The generic situation in Case 1.

a translation, represent the boundary by the constant zero function.

Case 2: $x \in I_\eta(r)$ **with** $|x'| < r - \tilde{r}_0$**.**

This case is similar to Case 1. Here, we can represent the boundary of $D_\eta(r)$ in $\{x\} + \widetilde{D}(\tilde{r}_0)$ by the function η and we find that the top of $\{x\} + \widetilde{D}(\tilde{r}_0)$ lies below the top of $D_\eta(r)$ if $\tilde{r}_0$ and $\widetilde{M}$ satisfy (1.17).

Case 3: $x \in B$ **with** $rM + \tilde{r}_0 < x_d < 10d(M+1)r - \tilde{r}_0$**.**

First, rotate $D_\eta(r)$ by 90 degrees in the x_1-x_d-plane in the mathematically positive direction, so that the rotated Lipschitz cylinder $RD_\eta(r)$ is "above" Rx, where R is given by $Ry = (y_d, y_2, \dots y_{d-1}, -y_1)$. Secondly, note that the condition on x_d considered in this case implies that the portion of the cylinder $\{Rx\} + \widetilde{D}(\tilde{r}_0)$ inside $RD_\eta(r)$ does neither intersect RT nor $RI_\eta(r)$.

Furthermore, the top of the cylinder $\widetilde{D}(\tilde{r}_0)$ is contained in $RD_\eta(r)$ if and only if the set

$$\{Rx + (y', 10d(\widetilde{M}+1)\tilde{r}_0) : |y'| < \tilde{r}_0\}$$

is contained in the infinite cylinder $\{|(y'', y_d)| < r\}$, and since the dth component of Rx is $-r$ this is the case if and only if

$$[10d(\widetilde{M}+1)\tilde{r}_0 - r]^2 + |y''|^2 < r^2$$

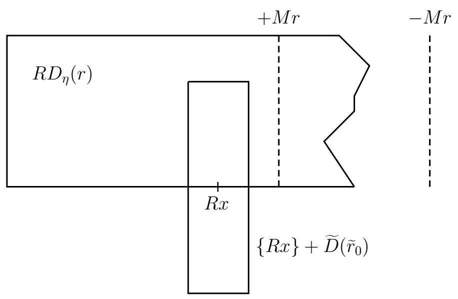

Figure 3: A Lipschitz cylinder that is rotated by 90 degrees in the x_1-x_d-plane.

for all y'' with $|(y_1, y'')| < \widetilde{r}_0$ for some y_1. Since $|y''|^2 < \widetilde{r}_0^2$, the inequality above is satisfied if $\widetilde{r}_0$ and $\widetilde{M}$ fulfill the condition

$$\frac{[10d(\widetilde{M}+1)]^2 + 1}{20d(\widetilde{M}+1)}\widetilde{r}_0 \leq r. \tag{1.18}$$

Next, we derive an expression for the graph that represents the boundary around x. A point satisfies $y \in R\partial D_\eta(r)$ with $y \in \{x\} + \widetilde{D}(\widetilde{r}_0)$ if and only if

$$|y''|^2 + |y_d|^2 = r^2$$

and

$$|y' - x_d e_1|^2 < \widetilde{r}_0^2.$$

Thus, the boundary is described by the function

$$\widetilde{\eta}_x(y') := -\sqrt{r^2 - |y''|^2}.$$

Here, we took the negative sign for the root of y_d, because we describe the boundary of $RD_\eta(r)$ around the point $Rx = -x_d e_1 - r e_d$. By (1.18) we infer that $\widetilde{r}_0 \leq r/(5d)$, so that the derivative of $\widetilde{\eta}_x$ is bounded on $B'(-x_d e_1, \widetilde{r}_0)$ by

$$\frac{1}{\sqrt{24d^2 - 1}}.$$

The same holds true for the Lipschitz constant of $\widetilde{\eta}_x$.
Case 4: $x \in B$ **with** $\eta(x') \leq x_d \leq Mr+\widetilde{r}_0$ **or** $x \in I_\eta(r)$ **with** $|x'| \geq r-\widetilde{r}_0$**.**

We rotate the Lipschitz cylinder $D_\eta(r)$ in the x_1-x_d-plane about the angle $\alpha := \arctan(1/(2M))$. Denote this rotation by R_α. Then, by Lemma 1.3.23 the rotated graph of η stays the graph of a Lipschitz function with Lipschitz constant depending only on M. With the right choice of $\widetilde{r}_0$ and $\widetilde{M}$ we have to ensure that the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ lies inside $R_\alpha D_\eta(r)$. To show that it lies "underneath" (with respect to the x_d-coordinate) the rotated top of $D_\eta(r)$, note that the x_d-coordinate of $R_\alpha y$ for $y \in T$ is given by

$$y_1 \sin(\alpha) + 10d(M+1)r\cos(\alpha). \tag{1.19}$$

This becomes minimal if y_1 is minimal. Moreover, only points in $R_\alpha T$ are of interest, that lie above the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$, i.e., that satisfy

$$|(R_\alpha y)' - (R_\alpha x)'| < \widetilde{r}_0, \tag{1.20}$$

see Figure 4 for a picture of the situation. A geometric consideration reveals that y_1 is minimal if $y'' = 0$. Thus, we can compute this minimal y_1 by making (1.20) an equality and by setting $y'' = 0$ as

$$(y_1 - x_1)\cos(\alpha) - (10d(M+1)r - x_d)\sin(\alpha) = -\widetilde{r}_0.$$

Consequently, by virtue of (1.19), we can estimate the lowest height of the part of $R_\alpha T$, which lies above the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ by

$$\begin{aligned} y_1 &\sin(\alpha) + 10d(M+1)r\cos(\alpha) \\ &\geq \frac{-\widetilde{r}_0 + x_1\cos(\alpha) + (10d(M+1)r - x_d)\sin(\alpha)}{\cos(\alpha)}\sin(\alpha) \\ &\quad + 10d(M+1)r\cos(\alpha) \\ &= \frac{(-\widetilde{r}_0 + x_1\cos(\alpha))\sin(\alpha) + 10d(M+1)r - x_d(1-\cos(\alpha)^2)}{\cos(\alpha)}. \end{aligned}$$

Now, the x_d-coordinate of any point in the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ is given by

$$x_1\sin(\alpha) + x_d\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0,$$

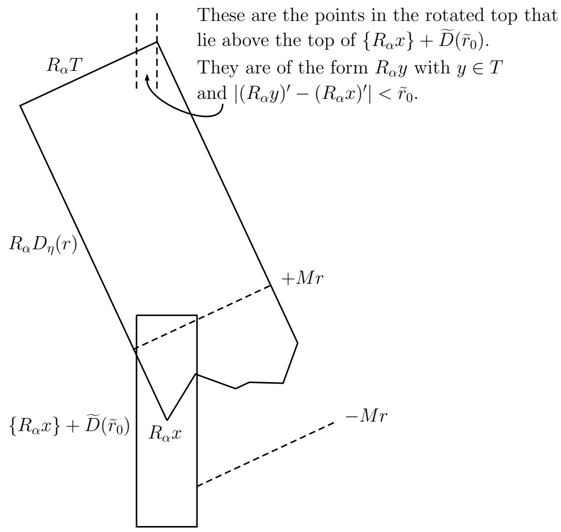

Figure 4: A Lipschitz cylinder that is rotated by the angle α together with the cylinder $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ and a scetch of the points in the rotated top that lie above the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$.

so that the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ is "underneath" $R_\alpha T$ if

$$\begin{aligned}
&x_1 \sin(\alpha) + x_d \cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0 \\
&\qquad \leq \frac{(-\widetilde{r}_0 + x_1\cos(\alpha))\sin(\alpha) + 10d(M+1)r - x_d(1-\cos(\alpha)^2)}{\cos(\alpha)} \\
&\qquad = -\widetilde{r}_0 \tan(\alpha) + x_1 \sin(\alpha) + \frac{10d(M+1)r}{\cos(\alpha)} - \frac{x_d}{\cos(\alpha)} + x_d\cos(\alpha).
\end{aligned}$$

Canceling equal terms on both sides of the last inequality shows that

$$[10d(\widetilde{M}+1) + \tan(\alpha)]\widetilde{r}_0 \leq \frac{10d(M+1)r}{\cos(\alpha)} - \frac{x_d}{\cos(\alpha)}.$$

To obtain a condition on $\widetilde{r}_0$ and $\widetilde{M}$ that is uniform in x, note that x_d is maximal if $x_d = Mr + \widetilde{r}_0$ (this follows by recalling the conditions imposed on points x considered in this case) so that the condition above turns into

$$[10d(\widetilde{M}+1) + \tan(\alpha)]\widetilde{r}_0 \leq \frac{10d(M+1)r}{\cos(\alpha)} - \frac{Mr}{\cos(\alpha)} - \frac{\widetilde{r}_0}{\cos(\alpha)}. \tag{1.21}$$

To see that the top of $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ does not intersect the cylinder barrel $R_\alpha B$, we rotate everything back (by the angle $-\alpha$), and conclude that the top does not intersect the rotated barrel if and only if

$$\{x\} + \left\{ R_{-\alpha} \begin{pmatrix} y' \\ 10d(\widetilde{M}+1)\widetilde{r}_0 \end{pmatrix} : |y'| < \widetilde{r}_0 \right\}$$

does not intersect B. In other words, if and only if

$$[x_1 + y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + |y''|^2 < r^2 \tag{1.22}$$

for all y' with $|y'| < \widetilde{r}_0$. In order to ensure (1.22), choose $\widetilde{M}$ large enough, such that

$$10d(\widetilde{M}+1)\tan(\alpha) > 1. \tag{1.23}$$

Since $|y_1| < \widetilde{r}_0$ this implies that

$$y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha) > 0.$$

Using this together with $-r \le x_1 \le \widetilde{r}_0 - r < 0$ and $|y''|^2 \le \widetilde{r}_0^2 - |y_1|^2$, we infer

$$\begin{aligned}
&[x_1 + y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + |y''|^2 \\
&\quad = x_1^2 + 2x_1[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)] \\
&\qquad\quad + [y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + |y''|^2 \\
&\quad \le r^2 + 2(\widetilde{r}_0 - r)[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)] \\
&\qquad\quad + [y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + \widetilde{r}_0^2 - |y_1|^2 .
\end{aligned}$$

The binomial formula together with $|y_1|^2\,[\cos(\alpha)^2 - 1] \le 0$ finally yields

$$\begin{aligned}
&\le r^2 + 2(\widetilde{r}_0 - r)[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)] \\
&\qquad + 20y_1 d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)\cos(\alpha) + [10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + \widetilde{r}_0^2.
\end{aligned}$$

In order to conclude that (1.22) is valid, we need a condition on $\widetilde{r}_0$ and $\widetilde{M}$, such that the right-hand side of the preceding estimate is estimated uniformly for all $|y_1| < \widetilde{r}_0$ by r^2. Thus, $\widetilde{r}_0$ and $\widetilde{M}$ have to satisfy the inequality

$$\begin{aligned}2(\widetilde{r}_0 - r)&[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]\\ &+ 20y_1 d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)\cos(\alpha) + [10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + \widetilde{r}_0^2 < 0.\end{aligned}$$

Rearranging yields

$$\begin{aligned}&2\widetilde{r}_0[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]\\ &\qquad + 20y_1 d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)\cos(\alpha) + [10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)]^2 + \widetilde{r}_0^2\\ &< 2r[y_1\cos(\alpha) + 10d(\widetilde{M}+1)\widetilde{r}_0\sin(\alpha)].\end{aligned}$$

Now, the condition is uniform for $\widetilde{r}_0$ and $\widetilde{M}$ if y_1 is replaced by $\widetilde{r}_0$ on the left-hand side of the inequality and if y_1 is replaced by $-\widetilde{r}_0$ on the right-hand side of the inequality. This yields

$$\begin{aligned}&\Big[1 + 2\cos(\alpha) + 20d(\widetilde{M}+1)\sin(\alpha)(1+\cos(\alpha)) + [10d(\widetilde{M}+1)\sin(\alpha)]^2\Big]\widetilde{r}_0^2\\ &\qquad < \Big[20d(\widetilde{M}+1)\sin(\alpha) - 2\cos(\alpha)\Big] r\widetilde{r}_0.\end{aligned} \tag{1.24}$$

Next, we identify the graph, which describes the intersection of the sets $\{R_\alpha x\} + \widetilde{D}(\widetilde{r}_0)$ and $R_\alpha \partial D_\eta(r)$. Since by Lemma 1.3.23 points on $R_\alpha I_\eta(r)$ can be described as a Lipschitz graph with Lipschitz constant depending only on M (by our choice of α), we concentrate on points of the form $R_\alpha y$ with $y \in B$ and with $-r \le y_1 \le -r/4$. This portion of the rotated cylinder barrel is given by

$$\begin{aligned}\Big\{&(y_1\cos(\alpha) - y_d\sin(\alpha), y'', y_1\sin(\alpha) + y_d\cos(\alpha)) :\\ &- r \le y_1 \le -\frac{r}{4}, |y'| = r, \text{ and } \eta(y') \le y_d \le 10d(M+1)r\Big\}.\end{aligned} \tag{1.25}$$

In order to calculate the graph which parameterizes (1.25) one has to find a function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ that solves

$$\psi(y_1\cos(\alpha) - y_d\sin(\alpha), y'') = y_1\sin(\alpha) + y_d\cos(\alpha)$$

for $-r \leq y_1 \leq -\frac{r}{4}, |y'| = r$, and $\eta(y') \leq y_d \leq 10d(M+1)r$. Denoting the new "$y_1$"-variable, which is given by $y_1 \cos(\alpha) - y_d \sin(\alpha)$ by $\widetilde{y}_1$, we find that the condition above is equivalent to

$$\psi(\widetilde{y}_1, y'') = \widetilde{y}_1 \tan(\alpha) + \frac{y_d}{\cos(\alpha)}.$$

In order to eliminate y_d, note

$$r^2 = y_1^2 + |y''|^2 = \Big(\frac{\widetilde{y}_1}{\cos(\alpha)} + y_d \tan(\alpha)\Big)^2 + |y''|^2 .$$

Solving this quadratic equation for y_d, one obtains

$$y_d = -\frac{\widetilde{y}_1}{\sin(\alpha)} \pm \left[\frac{r^2 - |y''|^2}{\tan(\alpha)^2}\right]^{\frac{1}{2}}. \tag{1.26}$$

Since by definition of $\widetilde{y}_1$

$$y_d = -\frac{\widetilde{y}_1}{\sin(\alpha)} + \frac{y_1}{\tan(\alpha)},$$

we find by $\alpha \in (0, \pi/2)$ and the negativity of y_1 that $y_d < -\widetilde{y}_1 / \sin(\alpha)$ and thus that

$$y_d = -\frac{\widetilde{y}_1}{\sin(\alpha)} - \left[\frac{r^2 - |y''|^2}{\tan(\alpha)^2}\right]^{\frac{1}{2}}.$$

This yields an expression for ψ, namely

$$\psi(\widetilde{y}_1, y'') = \widetilde{y}_1 \tan(\alpha) - \frac{\widetilde{y}_1}{\sin(\alpha)\cos(\alpha)} - \frac{1}{\cos(\alpha)}\left[\frac{r^2 - |y''|^2}{\tan(\alpha)^2}\right]^{\frac{1}{2}}. \tag{1.27}$$

This function is Lipschitz continuous because $r^2 - |y''|^2 = y_1^2 \geq r^2/16$ with Lipschitz constant depending only on α, and since α depends on M, only on M.

We claim that the projection of (1.25) to the $\{x_d = 0\}$-hyperplane yields

$$\mathcal{Z} := \Big\{(\widetilde{y}_1, y'') : |y''|^2 \leq \frac{15r^2}{16} \text{ and } (\bigstar)\Big\}$$

where (★) is given by the conditions

$$-\cos(\alpha)[r^2 - |y''|^2]^{\frac{1}{2}} - \sin(\alpha)10d(M+1)r < \widetilde{y}_1$$

and

$$\widetilde{y}_1 < -\cos(\alpha)[r^2 - |y''|^2]^{\frac{1}{2}} - \sin(\alpha)\eta\Big(-[r^2 - |y''|^2]^{\frac{1}{2}}, y''\Big) =: \vartheta(y'').$$

Indeed, if $(\widetilde{y}_1, y'', y_d)$ is an element of the set given in (1.25), its projection to the $\{x_d = 0\}$-hyperplane is given by $(\widetilde{y}_1, y'')$. Moreover, there are y_1 and y_d satisfying the conditions in (1.25) and $\widetilde{y}_1 = y_1 \cos(\alpha) - y_d \sin(\alpha)$. Note that $|(y_1, y'')| = r$ and $-r \leq y_1 \leq -r/4$ imply that $y_1 = -[r^2 - |y''|]^{1/2}$ and that $|y''|^2 \leq 15r^2/16$. Finally, (★) is satisfied due to the identities for $\widetilde{y}_1$ and y_1 together with $\eta(y_1, y'') \leq y_d \leq 10d(M+1)r$.

To prove the other inclusion, let $(\widetilde{y}_1, y'') \in \mathcal{Z}$ and define $y_1 := -[r^2 - |y''|]^{1/2}$. With this definition, we directly find $-r \leq y_1 \leq -r/4$ and $|(y_1, y'')| = r$. Next, define y_d via Equation (1.26). This definition shows that $\widetilde{y}_1$ is given by $y_1 \cos(\alpha) - y_d \sin(\alpha)$. Plugging this expression for $\widetilde{y}_1$ into (★), we directly infer that $\eta(y_1, y'') \leq y_d \leq 10d(M+1)r$. This shows that $(\widetilde{y}_1, y'')$ lies inside the projection of (1.25) onto the $\{x_d = 0\}$-hyperplane.

Notice the function ϑ appearing in the second condition of (★). Because η is Lipschitz continuous, it is clear that ϑ is Lipschitz continuous if $d \geq 3$. If $d = 2$, then $\mathcal{Z}$ is simply a line.

If x is again a point satisfying one of the conditions of the case, consider the set $B'((R_\alpha x)', \widetilde{r}_0)$. Since $(R_\alpha x)' = [x_1 \cos(\alpha) - x_d \sin(\alpha)]e_1$, we see that for $y' \in B'((R_\alpha x)', \widetilde{r}_0)$, we have $|y''| < \widetilde{r}_0$. By (1.18), we infer as in Case 3, that $\widetilde{r}_0 \leq r/(5d)$ and thus, that $B'((R_\alpha x)', \widetilde{r}_0)$ lies in the infinite cylinder

$$\Big\{(\widetilde{y}_1, y'') : |y''|^2 \leq \frac{15r^2}{16}\Big\}.$$

Next, apply Lemma 1.3.24 with $A_1 := \mathcal{Z} \cap \overline{B'((R_\alpha x)', \widetilde{r}_0)}$ and $\eta_1 := \psi$, cf. (1.27), and $A_2 := \overline{B'((R_\alpha x)', \widetilde{r}_0)} \setminus A_1$ and η_2 being the Lipschitz function parameterizing the rotated graph of η. This "glued" function $\widetilde{\eta}_x$ is Lipschitz continuous and has a Lipschitz constant, which depends only on M and d.

Case 5: $x \in B$ with $x_d \geq 10d(M+1)r - \widetilde{r}_0$ or $x \in T$ with $|x'| \geq r - \widetilde{r}_0$.

This case is a special case of Case 4, because up to a rotation about 180 degrees in the x_1-x_d-plane, and up to a translation, the top can by viewed as the graph of the zero function. In this case, one should read Case 4 with $M = 0$.

Conclusion. Choose $\widetilde{M}$ to be the maximum of all Lipschitz constants occurred in the proof and such that additionally (1.23) holds. Then $\widetilde{M}$ is a constant that depends only on M and d. Choose $\widetilde{r}_0$ small enough such that (1.17), (1.18), and (1.21) are valid. Then $\widetilde{r}_0$ depends only on M, d, and r. We see that all these inequalities for $\widetilde{r}_0$ bound this quantity always linearly by r, so that one can find a uniform $\widetilde{r}_0$ depending only on r' for all Lipschitz cylinders $D_\eta(r)$ with $r \geq r'$. This concludes this very technical proof. □

Remark 1.3.27. Considering a family of Lipschitz cylinders $D_\eta(s)$ with $1 \leq s \leq 2$, we see by the previous lemma that one can cover $\partial D_\eta(s)$ for each $s \in [1, 2]$ by the same number of cylinders and that in each of the cylinders the Lipschitz constant of the function that describes the boundary of $D_\eta(s)$ inside the cylinder has a Lipschitz constant depending only on M and d. By definition of the Lipschitz character, we conclude that the sets $D_\eta(s)$, $s \in [1, 2]$, all have the same Lipschitz character.

In the following proposition, we present the application of the argument involving the co-area formula as discussed before Theorem 1.3.22. This proposition was used implicitly by Shen in [87, Lem. 4.1]. However, since we could not find a proof, we present one here.

Proposition 1.3.28. *Let $1 \leq p < \infty$, $u \in \mathrm{W}^{1,p}(D_\eta(3))$, $\mathfrak{u}$ be a representative of u with $(\mathfrak{u})^* \in \mathrm{L}^p(I_\eta(12/5))$ and $\mathfrak{u} \to 0$ non-tangentially σ-a.e. on $I_\eta(12/5)$, and $\nabla\mathfrak{u}$ be a representative of ∇u. Furthermore, assume that each regular cone $\Gamma(q)$ with $q \in I_\eta(12/5)$ that defines the non-tangential maximal function $()^*$ contains the axis parallel to the x_d-axis. Then there exists a set $\mathcal{N} \subset (0, 12/5)$ of measure zero, such that*

$$h(x) := \begin{cases} \mathfrak{u}(x), & \text{for } x \in \partial D_\eta(r) \setminus I_\eta(r) \\ 0, & \text{for } x \in I_\eta(r) \end{cases}$$

defines an element of $\mathrm{W}^{1,p}(\partial D_\eta(r))$ for every $r \in (0, 12/5) \setminus \mathcal{N}$. Moreover, there exists a constant $C > 0$ depending only on d and M, such that

$$\|\nabla_{\tan} h\|_{\mathrm{L}^p(\partial D_\eta(r); \mathbb{C}^d)} \leq C \|\nabla\mathfrak{u}\|_{\mathrm{L}^p(\partial D_\eta(r) \setminus I_\eta(r); \mathbb{C}^d)} < \infty.$$

Proof. Step 1: The construction of $\mathcal{N}$.

Since $u \in \mathrm{W}^{1,p}(D_\eta(3))$ and since $D_\eta(3)$ is a Lipschitz domain, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty(\overline{D_\eta(3)})$ with $\psi_n \to u$ in $\mathrm{W}^{1,p}(D_\eta(3))$, see McLean [70, Thm. 3.29]. Define

$$f : \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto t \quad \text{iff} \quad x \in \partial D(t)$$

and $f(0) = 0$. Since for $x \in \partial D(t)$ and $y \in \partial D(s)$ we have

$$|f(x) - f(y)| = |t - s| \leq |x - y|\,,$$

so that the function f is Lipschitzian with Lipschitz constant coinciding with 1. Note that the surface measure σ on $\partial D(r)$ and the $(d-1)$-dimensional Hausdorff measure on $\partial D(r)$ are comparable with implicit constants depending only on d and the Lipschitz constant M (this follows by comparing m_{d-1} with the $(d-1)$-dimensional Lebesgue measure and by using (1.6)). Then, applying the co-area formula Theorem 1.3.22 twice with g being $|u - \psi_n|^p \chi_{D_\eta(3)}$ and $|\nabla u - \nabla \psi_n|^p \chi_{D_\eta(3)}$, respectively, yields

$$\begin{aligned}
&\int_0^3 \int_{\partial D_\eta(r)\setminus I_\eta(r)} |\mathfrak{u}(x) - \psi_n(x)|^p \,\mathrm{d}\sigma(x)\,\mathrm{d}r \\
&\qquad + \int_0^3 \int_{\partial D_\eta(r)\setminus I_\eta(r)} |\nabla \mathfrak{u}(x) - \nabla \psi_n(x)|^p \,\mathrm{d}\sigma(x)\,\mathrm{d}r \\
&\leq C\biggl\{ \int_{D_\eta(3)} |\mathfrak{u}(x) - \psi_n(x)|^p \,\mathrm{d}x + \int_{D_\eta(3)} |\nabla \mathfrak{u}(x) - \nabla \psi_n(x)|^p \,\mathrm{d}x \biggr\},
\end{aligned}$$

where the right-hand side converges to zero as $n \to \infty$ and where C solely depends on d and M. We deduce that there exists a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ of $(\psi_n)_{n\in\mathbb{N}}$ and a set of measure zero $\mathcal{K} \subset (0,3)$, such that for every $r \in (0,3) \setminus \mathcal{K}$ the convergences

$$\begin{aligned}
\int_{\partial D_\eta(r)\setminus I_\eta(r)} |\mathfrak{u}(x) - \psi_{n_k}(x)|^p \,\mathrm{d}\sigma(x) \to 0 \quad \text{as} \quad k \to \infty \\
\int_{\partial D_\eta(r)\setminus I_\eta(r)} |\nabla \mathfrak{u}(x) - \nabla \psi_{n_k}(x)|^p \,\mathrm{d}\sigma(x) \to 0 \quad \text{as} \quad k \to \infty
\end{aligned}$$

hold. It follows that $\mathfrak{u} \in \mathrm{L}^p(\partial D_\eta(r)\setminus I_\eta(r))$ and $\nabla \mathfrak{u} \in \mathrm{L}^p(\partial D_\eta(r)\setminus I_\eta(r); \mathbb{C}^d)$ for every $r \in (0,3) \setminus \mathcal{K}$.

Let $\mathcal{L} \subset I_\eta(12/5)$ denote the set of all points, where the non-tangential limit of $\mathfrak{u}$ on $I_\eta(12/5)$ does not exist. By assumption this set has surface measure zero. This together with (1.6) and the co-area formula with $f : \mathbb{R}^{d-1} \to \mathbb{R}, x' \mapsto |x'|$ yields in the case $d \geq 3$

$$\begin{aligned}\int_0^{12/5} \int_{\partial B'(0,r)} &\chi_{\mathcal{L}}(x', \eta(x'))[1 + |\nabla_{x'}\eta(x')|^2]^{\frac{1}{2}} \, \mathrm{d}\sigma_{B'_r}(x') \, \mathrm{d}r \\ &\leq C \int_{B'(0,12/5)} \chi_{\mathcal{L}}(x', \eta(x'))[1 + |\nabla_{x'}\eta(x')|^2]^{\frac{1}{2}} \, \mathrm{d}x' \\ &= 0,\end{aligned}$$

where $\sigma_{B'_r}$ denotes the surface measure on $\partial B'(0, r)$. Moreover, because $(\mathfrak{u})^* \in \mathrm{L}^p(I_\eta(12/5))$ the same integration argument yields

$$\int_0^{12/5} \int_{\partial B'(0,r)} |(\mathfrak{u})^*(x', \eta(x'))|^p [1 + |\nabla_{x'}\eta(x')|^2]^{\frac{1}{2}} \, \mathrm{d}\sigma_{B'_r}(x') \, \mathrm{d}r < \infty.$$

This implies the existence of a set of measure zero $\mathcal{M} \subset (0, 12/5)$ with the properties, that

$$\sigma_{B'_r}(\{x' \in \mathbb{R}^{d-1} : |x'| = r \text{ and } (x', \eta(x')) \in \mathcal{L}\}) = 0$$

and $(\mathfrak{u})^*(\cdot, \eta(\cdot)) \in \mathrm{L}^p(\partial B'(0, r))$ for every $r \in (0, 12/5) \setminus \mathcal{M}$.

In the case $d = 2$, let $\mathcal{M}$ be the union of $\mathcal{L}$ and the set of all points, where $(\mathfrak{u})^*$ is infinite.

Finally, define $\mathcal{N} := [\mathcal{K} \cup \mathcal{M}] \cap (0, 12/5)$.

In the following, r will always be an element of $(0, 12/5) \setminus \mathcal{N}$.

Step 2: Weak differentiability in cylinders intersecting only $I_\eta(r)$ or $\partial D_\eta(r) \setminus I_\eta(r)$.

Let Φ_{x,r_0} be the coordinate function corresponding to the set U_{x,r_0}, cf. (1.5). If U_{x,r_0} intersects only $I_\eta(r)$, then, by definition, h is constantly zero and hence weakly differentiable with weak derivative in $\mathrm{L}^p(I_\eta(r))$.

If U_{x,r_0} intersects only $\partial D_\eta(r) \setminus I_\eta(r)$, one finds for the approximating sequence $(\psi_{n_k})_{k\in\mathbb{N}}$ from Step 1 and for $\varphi \in \mathrm{C}_c^\infty(B'(0, r_0))$

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} &\partial_i\varphi(y')[\psi_{n_k} \circ \Phi_{x,r_0}](y') \, \mathrm{d}y' \\ &= -\int_{\mathbb{R}^{d-1}} \varphi(y')\langle[\nabla\psi_{n_k}](\Phi_{x,r_0}(y')), \partial_i\Phi_{x,r_0}(y')\rangle \, \mathrm{d}y',\end{aligned}$$

whenever $i = 1, \dots, d-1$. Due to $r \notin \mathcal{N}$ and the properties derived in Step 1, one finds in the limit $k \to \infty$

$$\int_{\mathbb{R}^{d-1}} \partial_i \varphi(y')[\mathfrak{u} \circ \Phi_{x,r_0}](y') \, \mathrm{d}y' \\ = - \int_{\mathbb{R}^{d-1}} \varphi(y') \langle [\nabla \mathfrak{u}](\Phi_{x,r_0}(y')), \partial_i \Phi_{x,r_0}(y') \rangle \, \mathrm{d}y'.$$

Consequently, the weak derivatives of h exist in $U_{x,r_0} \cap [\partial D_\eta(r) \setminus I_\eta(r)]$ and lie in $\mathrm{L}^p(\partial D_\eta(r) \setminus I_\eta(r))$.

Step 3: Weak differentiability in cylinders intersecting both $I_\eta(r)$ and $\partial D_\eta(r) \setminus I_\eta(r)$.

Let $x = (y', \eta(y'))$ for some $|y'| = r$ and let Φ_{x,r_0} be the corresponding coordinate function with r_0 being the number from Lemma 1.3.25. The same lemma implies that the portion of the boundary, which comes from the cylinder barrel is given by

$$\mathcal{Z} := \{(w_1, w'') : w_1 \le \vartheta(w''), |(w_1, w'')| < r_0\}$$

for a Lipschitz function ϑ defined on $\{|w''|^2 < 15r^2/16\}$. Moreover, define

$$\mathcal{Z}_n := \{(w_1, w'') : w_1 \le \vartheta(y'') - 1/n, |(w_1, w'')| < r_0\}.$$

In order to show that h is weakly differentiable on $U_{x,r_0} \cap \partial D_\eta(r)$, take $\varphi \in \mathrm{C}_c^\infty(B'(0, r_0))$. Due to $h = 0$ on $I_\eta(r)$ one calculates

$$\int_{\mathbb{R}^{d-1}} \partial_i \varphi(w')[h \circ \Phi_{x,r_0}](w') \, \mathrm{d}w' = \int_{\mathcal{Z}} \partial_i \varphi(w')[\mathfrak{u} \circ \Phi_{x,r_0}](w') \, \mathrm{d}w'.$$

By the choice of r, $\mathfrak{u}$ is integrable on $\partial D_\eta(r) \setminus I_\eta(r)$ so that Lebesgue's dominated convergence theorem yields

$$= \lim_{n \to \infty} \int_{\mathcal{Z}_n} \partial_i \varphi(w')[\mathfrak{u} \circ \Phi_{x,r_0}](w') \, \mathrm{d}w'$$

and integration by parts, cf. Ziemer [102, Thm. 5.8.2, Rem. 5.8.3] together with the boundary conditions of φ give that the integral coincides with

$$- \int_{\mathcal{Z}_n} \varphi(w') \Big\langle [\nabla \mathfrak{u}](\Phi_{x,r_0}(w')), \partial_i \Phi_{x,r_0}(w') \Big\rangle \, \mathrm{d}w' \\ + \int_{\{|y''| < r_0\}} \Bigg\{ [\mathfrak{u} \circ \Phi_{x,r_0}]\Big(\vartheta(w'') - 1/n, w''\Big) \\ \cdot \varphi\Big(\vartheta(w'') - 1/n, w''\Big) \frac{(1, \nabla_{w''} \vartheta(w''))_i}{\sqrt{1 + |\nabla_{w''} \vartheta(w'')|^2}} \Bigg\} \, \mathrm{d}w'',$$

where $(1, \nabla_{w''}\vartheta(w''))_i$ is the ith component of the normal vector to the Lipschitz graph of $\vartheta - 1/n$. By the choice of r, we have $\nabla\mathfrak{u} \in \mathrm{L}^p(\partial D_\eta(r) \setminus I_\eta(r); \mathbb{C}^d)$, so that by the dominated convergence theorem the first integral of this last expression converges to

$$\int_{\mathcal{Z}} \varphi(w') \Big\langle [\nabla\mathfrak{u}](\Phi_{x,r_0}(w')), \partial_i \Phi_{x,r_0}(w') \Big\rangle \, \mathrm{d}w'.$$

The second integral converges to zero, because on the one hand,

$$[\mathfrak{u} \circ \Phi_{x,r_0}]\Big(\vartheta(w'') - 1/n, w''\Big) \to 0 \quad \text{as} \quad n \to \infty$$

for almost every w'', because r was chosen such that the non-tangential limit of $\mathfrak{u}$ to $\{(y', \eta(y')) : |y'| = r\}$ exists almost everywhere with regard to the measure $\sigma_{B'_r}$ and because the limit is zero on $I_\eta(\frac{12}{5})$. On the other hand, Lebesgue's dominated convergence theorem is applicable, because r is chosen such that $(\mathfrak{u})^*(\cdot, \eta(\cdot)) \in \mathrm{L}^p(\partial B'(0, r))$. This finally proves $h \in \mathrm{W}^{1,p}(\partial D_\eta(r))$. $\square$

CHAPTER 2

A short glimpse into operator theory

This chapter is intended to serve as a short introduction to basic notions of operator theory and maximal L^q-regularity. We begin this chapter with a brief introduction to sectorial operators, functional calculus, and analytic semigroups. Here, we rely on the books of Haase [46] and Engel and Nagel [25]. Then, we will turn to the notion of maximal L^q-regularity. We summarize basic properties of operators with maximal L^q-regularity, in particular, we emphasize a characterization of maximal L^q-regularity of operators on subspaces of L^p via square function estimates. This will be fundamental in the study of the Stokes operator and higher-order elliptic systems in Chapters 5 and 7. For general introductions to maximal L^q-regularity we refer to Amann [3], Denk, Hieber, and Prüss [18], and Kunstmann and Weis [61].

2.1 Sectorial operators, functional calculus, and analytic semigroups

For $\theta \in (0, \pi)$ define the sector in the complex plane

$$\mathrm{S}_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$$

and for $\theta = 0$ define $\mathrm{S}_\theta := (0, \infty)$.

Definition 2.1.1. A closed linear operator A on a Banach space X is called *sectorial of angle* ω if there exists $\omega \in [0, \pi)$ such that $\sigma(A) \subset \overline{\mathrm{S}_\omega}$ and if for every $\theta \in (\omega, \pi]$ there exists a constant $C > 0$ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C \qquad (\lambda \in \mathrm{S}_{\pi-\theta}).$$

Remark 2.1.2. A sectorial operator on a reflexive Banach space X is densely defined, see Haase [46, Prop. 2.1.1].

To define a functional calculus for a sectorial operator fix $\vartheta \in (0, \pi)$ and consider the *Dunford-Riesz class* $\mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ consisting of all holomorphic functions $f : \mathrm{S}_\vartheta \to \mathbb{C}$ satisfying for some $C, \varepsilon > 0$ the inequality

$$|f(z)| \leq C \frac{|z|^\varepsilon}{1 + |z|^{2\varepsilon}} \qquad (z \in \mathrm{S}_\vartheta).$$

For a sectorial operator A of angle ω and for every $\vartheta \in (\omega, \pi)$ one can define the expression $f(A)$ for each $f \in \mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ via the Cauchy integral

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)(z - A)^{-1} \,\mathrm{d}z,$$

where γ denotes the path which rounds $\partial \mathrm{S}_\varphi$ for some $\varphi \in (\omega, \vartheta)$ counterclockwise. The resolvent estimate in Definition 2.1.1 implies that the integral converges in the topology of $\mathcal{L}(X)$ and thus defines a bounded linear operator. If $\lambda \notin \overline{\mathrm{S}_\vartheta}$ one directly shows that $z \mapsto (\lambda - z)^{-1} f(z) \in \mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ for every $f \in \mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ and one can verify that

$$(\lambda - A)^{-1} f(A) = [(\lambda - \cdot)^{-1} f(\cdot)](A),$$

which gives rise to an extension of the functional calculus to a larger algebra of functions, namely to the *extended Dunford-Riesz class* $\mathcal{E}(\mathrm{S}_\vartheta)$ which consists of the algebra generated by $\mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$, the function $z \mapsto (1 + z)^{-1}$, and by the function constant to one. Thus, for any $f \in \mathcal{E}(\mathrm{S}_\vartheta)$ one can find complex numbers a, b and a function $f_0 \in \mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ such that

$$f(z) = f_0(z) + a(1 + z)^{-1} + b,$$

and one defines

$$f(A) := f_0(A) + a(1+A)^{-1} + b\,\mathrm{Id} \in \mathcal{L}(X).$$

One can prove that the operator Φ_A, which maps $f \in \mathcal{E}(\mathrm{S}_\vartheta)$ to $f(A)$ is an algebra homomorphism, see Haase [46, Thm. 2.3.3]. In order to extend the algebra homomorphism to still larger algebras one proceeds by *regularization*. Here, if $f : \mathrm{S}_\vartheta \to \mathbb{C}$ is holomorphic, one needs an element $e \in \mathcal{E}(\mathrm{S}_\theta)$ such that $e(A)$ is injective and such that $ef \in \mathcal{E}(\mathrm{S}_\vartheta)$. Then one can invert $e(A)$ and one can define the operator $f(A)$ by

$$\begin{aligned} \mathcal{D}(f(A)) &:= \{x \in X : [ef](A)x \in \mathcal{R}(e(A))\} \\ f(A) &:= e(A)^{-1}[ef](A). \end{aligned}$$

Using the regularizer $e(z) := (1+z)^{-k}$ for some fixed $k \in \mathbb{N}$ enables us to define $f(A)$ for all holomorphic functions on S_ϑ which have a polynomial limit at zero (i.e., there exists $c \in \mathbb{C}$ such that $f(z) - c = \mathcal{O}(|z|^\alpha)$ for some $\alpha > 0$ as $z \to 0$) and have polynomial growth at infinity. Particularly, one can define the fractional powers A^α if $\alpha \in \mathbb{C}$ has positive real part. In addition, if A is injective one can regularize with $e(z) := [z/(1+z)^2]^k$ for $k \in \mathbb{N}$ and one can define $f(A)$ for all f which have polynomial growth at zero and at infinity. In this case one can define A^α for all $\alpha \in \mathbb{C}$.

Let us assume for a moment that A is injective. An important algebra is the algebra of bounded holomorphic functions on the sector S_ϑ, denoted by $\mathrm{H}^\infty(\mathrm{S}_\vartheta)$, which can be obtained by regularizing with $e(z) := z/(1+z)^2$. Since $\mathrm{H}^\infty(\mathrm{S}_\vartheta)$ is not very far from the extended Dunford-Riesz class one might wonder whether $f(A)$ defines a bounded operator for every $f \in \mathrm{H}^\infty(\mathrm{S}_\vartheta)$. Unfortunately, this question has to be negated for general sectorial operators, see Haase [46, Sec. 9.1], but if a sectorial operator has the property that for every $f \in \mathrm{H}^\infty(\mathrm{S}_\vartheta)$ the operator $f(A)$ is bounded and if there exists a constant $C > 0$ such that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\vartheta)} \qquad (f \in \mathrm{H}^\infty(\mathrm{S}_\vartheta)),$$

then we say that the H^∞*-calculus of A is bounded*.

We close this section with analytic semigroups. Here, the operator A does not need to be injective. For a sectorial operator A of angle

$\omega \in [0, \pi/2)$ let $\vartheta \in (\omega, \pi/2)$ and $w \in \mathrm{S}_{\pi/2-\vartheta}$. In this situation, the function $z \mapsto \mathrm{e}^{-wz}$ is holomorphic on S_ϑ with $z \mapsto \mathrm{e}^{-wz} - 1 \in \mathrm{H}_0^\infty(\mathrm{S}_\vartheta)$ so that one can define e^{-wA} for every $w \in \mathrm{S}_{\pi/2-\vartheta}$. In the following, we will say that $-A$ is the *generator of a bounded analytic semigroup* if A is a densely defined, sectorial operator of angle $\omega \in [0, \pi/2)$. In this situation, one can also show that e^{-wA} has a representation as a Cauchy integral. Indeed,

$$\mathrm{e}^{-wA} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{wz} (z + A)^{-1} \, \mathrm{d}z \qquad (w \in \mathrm{S}_{\pi/2-\vartheta}), \tag{2.1}$$

where γ can be any of the paths parameterizing the boundary of $\mathrm{S}_\varphi \backslash B(0, r)$ for some $r > 0$ and $\varphi \in (\pi - \vartheta, \pi - \omega)$ counterclockwise. Further properties are:

(1) $\mathrm{e}^{-w_1 A} \mathrm{e}^{-w_2 A} = \mathrm{e}^{-(w_1 + w_2)A}$ for all $w_1, w_2 \in \mathrm{S}_{\pi/2-\vartheta}$;

(2) $\mathrm{e}^{-wA} x \to x$ whenever $w \to 0$ with $w \in \mathrm{S}_{\pi/2-\theta}$ and $\theta \in (\vartheta, \pi/2]$;

(3) the function $w \mapsto \mathrm{e}^{-wA}$ is analytic with respect to the operator norm and uniformly bounded on sectors strictly contained in $\mathrm{S}_{\pi/2-\vartheta}$.

For the properties above, consult Engel and Nagel [25, Sec. II.4].

2.2 Maximal L^q-regularity

Let in the following A be always such that $-A$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA})_{t \geq 0}$. Consider the *abstract Cauchy problem*

$$\text{(ACP)} \qquad \begin{cases} u'(t) + Au(t) = f(t), & t \geq 0 \\ \qquad\qquad u(0) = x, \end{cases}$$

where $f \in \mathrm{L}^q(0, \infty; X)$ and $x \in X$. A function $u \in \mathrm{C}([0, \infty); X)$ is called a *mild solution* of (ACP) if $\int_0^t u(s) \, \mathrm{d}s \in \mathcal{D}(A)$ for every $t \in [0, \infty)$ and if

$$u(t) = x - A \int_0^t u(s) \, \mathrm{d}s + \int_0^t f(s) \, \mathrm{d}s \qquad \text{for } t \geq 0$$

holds. It is classical, that there exists a unique mild solution for every $x \in X$ and $f \in \mathrm{L}^q(0,\infty;X)$, which is given by

$$u(t) = \mathrm{e}^{-tA}x + \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s \qquad t \geq 0, \tag{2.2}$$

see Arendt, Batty, Hieber, and Neubrander [5, Prop. 3.1.16]. Note that the existence and uniqueness in [5, Prop. 3.1.16] is only established for mild solutions on arbitrary bounded intervals $[0,\tau]$, $\tau > 0$. But if a unique mild solution exists on arbitrary intervals one can directly construct a unique "maximal" mild solution on $[0,\infty)$.

If the right-hand side of (ACP) lies in $\mathrm{L}^q(0,\infty;X)$, one could ask, whether or not each summand on the left-hand side lies in $\mathrm{L}^q(0,\infty;X)$ as well. This property is formalized by the following definition.

Definition 2.2.1 (Maximal L^q-regularity)**.** Let $-A$ be the generator of a bounded analytic semigroup on a Banach space X and $1 < q < \infty$. The operator A has *maximal* L^q*-regularity* if for $x = 0$ and all $f \in \mathrm{L}^q(0,\infty;X)$ the mild solution of (ACP) is differentiable a.e., $u(t) \in \mathcal{D}(A)$ for a.e. $t > 0$, and $u', Au \in \mathrm{L}^q(0,\infty;X)$.

Remark 2.2.2. The closed graph theorem implies that, for an operator A having maximal L^q-regularity, there exists a constant $C > 0$ possibly depending on q such that

$$\|u'\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \leq C\|f\|_{\mathrm{L}^q(0,\infty;X)} \tag{2.3}$$

holds for all $f \in \mathrm{L}^q(0,\infty;X)$ and corresponding mild solutions u.

The maximal regularity estimate (2.3) can also be extended to mild solutions with inhomogeneous initial data x in the real interpolation space $(X,\mathcal{D}(A))_{1-1/q,q}$.

Proposition 2.2.3. *If A has maximal* L^q*-regularity for some* $1 < q < \infty$*, then for all* $f \in \mathrm{L}^q(0,\infty;X)$ *and* $x \in (X,\mathcal{D}(A))_{1-1/q,q}$ *the corresponding mild solution u is differentiable a.e., satisfies* $u(t) \in \mathcal{D}(A)$ *for a.e.* $t > 0$*, and*

$$\|u'\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \leq C\Big\{\|f\|_{\mathrm{L}^q(0,\infty;X)} + \|x\|_{(X,\mathcal{D}(A))_{1-1/q,q}}\Big\}$$

with a constant $C > 0$ *depending only on the constants appearing in* (2.3) *and Proposition 1.2.7.*

Proof. Recall the trace method introduced in Subsection 1.2.2 and take $g \in \mathrm{V}(q, 1/q, \mathcal{D}(A), X)$ with $g(0) = x$ and $\|g\|_{\mathrm{V}(q,1/q,\mathcal{D}(A),X)} \le 2\|x\|^{\mathrm{Tr}}_{1-1/q,q}$. The existence of such a function g is provided by Proposition 1.2.7. By the definition of the norm of $V(q, 1/q, \mathcal{D}(A), X)$, see Definition 1.2.6, we deduce that $g, g', Ag \in \mathrm{L}^q(0, \infty; X)$. Let v be the mild solution to the problem

$$\begin{cases} v' + Av = f - g' - Ag, & t > 0 \\ \quad v(0) = 0. \end{cases}$$

Then, by uniqueness we find that $v + g$ coincides with u. Moreover, $g \in \mathrm{W}^{1,q}(0, \infty; X)$ implies that g is differentiable a.e., cf. [5, Prop. 1.2.2], and by virtue of the maximal regularity of A, v is differentiable a.e. and $v(t) \in \mathcal{D}(A)$ for a.e. $t > 0$. Consequently, the same holds true for u. Finally, (2.3) yields

$$\|u'\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \le C\Big\{\|f\|_{\mathrm{L}^q(0,\infty;X)} + \|g\|_{\mathrm{V}(q,1/q,\mathcal{D}(A),X)}\Big\}.$$

Since $\|x\|^{\mathrm{Tr}}_{1-1/q,q}$ is equivalent to $\|x\|_{(X,\mathcal{D}(A))_{1-1/q,q}}$ by Proposition 1.2.7, the special choice of g concludes the proof. □

2.3 Interconnection between $\mathcal{R}$-boundedness and maximal L^q-regularity

On Banach spaces of class $\mathcal{HT}$, see Definition 1.2.2, maximal L^q-regularity can be characterized by means of the $\mathcal{R}$-boundedness of the family of operators $\{\lambda(\lambda + A)^{-1} : \lambda \in \mathrm{S}_\theta\}$ for some $\theta > \pi/2$. This notion is introduced in the following definition.

Definition 2.3.1. A family $\mathcal{T} \subset \mathcal{L}(X)$ is said to be $\mathcal{R}$-*bounded* if there exists a constant $C > 0$ such that for all $n_0 \in \mathbb{N}$, $T_1, \dots, T_{n_0} \in \mathcal{T}$, and $x_1, \dots, x_{n_0} \in X$ the inequality

$$\Big\|\sum_{n=1}^{n_0} r_n T_n x_n\Big\|_{\mathrm{L}^2(0,1;X)} \le C\Big\|\sum_{n=1}^{n_0} r_n x_n\Big\|_{\mathrm{L}^2(0,1;X)}$$

holds. Here $r_n(t) := \mathrm{sgn}(\sin(2^n \pi t))$ are the *Rademacher functions.*

Remark 2.3.2. (1) If X is a Hilbert space the elements of each of the sets $\{r_1 x_1, \dots, r_{n_0} x_{n_0}\}$ and $\{r_1 T_1 x_1, \dots, r_{n_0} T_{n_0} x_{n_0}\}$ are pairwise orthogonal in $\mathrm{L}^2(0,1;X)$. This together with $\int_0^1 |r_n|^2 \, \mathrm{d}t = 1$ for each $n \in \mathbb{N}$ implies that a family of bounded operators $\mathcal{T}$ on a Hilbert space is bounded if and only if it is $\mathcal{R}$-bounded.

(2) An application of Kahane's inequality, see Diestel, Jarchow, and Tonge [23, Kahane's Ineq. 11.1], shows that one can replace the space $\mathrm{L}^2(0,1;X)$ in the definition of $\mathcal{R}$-boundedness by $\mathrm{L}^q(0,1;X)$ for each $1 \leq q < \infty$.

(3) Kalton and Weis proved in [57, Lem. 3.1] that if a family $\mathcal{T} \subset \mathcal{L}(X)$ is $\mathcal{R}$-bounded and X is a Banach space of non-trivial type, see [23, Sec. 11] for this notion, that then $\mathcal{T}^* \subset \mathcal{L}(X^*)$ is $\mathcal{R}$-bounded, where

$$\mathcal{T}^* := \{T^* : T \in \mathcal{T}\}$$

is the family of adjoint operators in $\mathcal{T}$ acting on the dual space X^* of X. Note that all closed subspaces of $\mathrm{L}^p(\Omega;\mathbb{C}^N)$ have non-trivial type whenever $1 < p < \infty$, see [23, Cor. 11.7].

(4) For $X = \mathrm{L}^p(\Omega)$, where $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ is Lebesgue measurable, one can reformulate $\mathcal{R}$-boundedness by means of a square function estimate, see, e.g., Kunstmann and Weis [61, Rem. 2.9]. More precisely, a family $\mathcal{T} \subset \mathcal{L}(X)$ is $\mathcal{R}$-bounded if and only if there exists a constant $C > 0$ such that for all $n_0 \in \mathbb{N}$, $T_1, \dots, T_{n_0} \in \mathcal{T}$, and $f_1, \dots, f_{n_0} \in \mathrm{L}^p(\Omega)$ the *square function estimate*

$$\Big\| \Big(\sum_{n=1}^{n_0} |T_n f_n|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\Omega)} \leq C \Big\| \Big(\sum_{n=1}^{n_0} |f_n|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\Omega)} \tag{2.4}$$

holds.

Proposition 2.3.3. *The statement of Remark 2.3.2 (4) remains valid if X is a closed subspace of $\mathrm{L}^p(\Omega;\mathbb{C}^N)$ for some $N \in \mathbb{N}$.*

Proof. It was proven in [61, Eq. (2.6)] that there exists a constant $C > 0$ depending only on p such that for all $g_1, \dots, g_{n_0} \in \mathrm{L}^p(\Omega)$

$$\frac{1}{C} \Big\| \sum_{n=1}^{n_0} r_n g_n \Big\|_{\mathrm{L}^p(0,1;\mathrm{L}^p(\Omega))} \leq \Big\| \Big(\sum_{n=1}^{n_0} |g_n|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\Omega)} \leq C \Big\| \sum_{n=1}^{n_0} r_n g_n \Big\|_{\mathrm{L}^p(0,1;\mathrm{L}^p(\Omega))}$$

holds. Note that by default, $\mathbb{C}^N$ is equipped with the Euclidean norm so that by equivalence of norms in $\mathbb{C}^N$, there exists a constant $C > 0$ depending only on N and p such that

$$\begin{aligned}\frac{1}{C}\Big\|\sum_{n=1}^{n_0} r_n T_n f_n\Big\|_{\mathrm{L}^p(0,1;X)} &\le \Big(\sum_{i=1}^{N}\Big\|\sum_{n=1}^{n_0} r_n \pi_i T_n f_n\Big\|^p_{\mathrm{L}^p(0,1;\mathrm{L}^p(\Omega))}\Big)^{\frac{1}{p}} \\ &\le C\Big\|\sum_{n=1}^{n_0} r_n T_n f_n\Big\|_{\mathrm{L}^p(0,1;X)},\end{aligned}$$

where π_i is the projection onto the ith component of a vector in $\mathbb{C}^N$. By virtue of the first statement of this proof we can estimate

$$\begin{aligned}\frac{1}{C}\Big(\sum_{i=1}^{N}\Big\|\sum_{n=1}^{n_0} r_n \pi_i T_n f_n\Big\|^p_{\mathrm{L}^p(0,1;\mathrm{L}^p(\Omega))}\Big)^{\frac{1}{p}} &\le \Big(\sum_{i=1}^{N}\Big\|\Big(\sum_{n=1}^{n_0} |\pi_i T_n f_n|^2\Big)^{\frac{1}{2}}\Big\|^p_{\mathrm{L}^p(\Omega)}\Big)^{\frac{1}{p}} \\ &\le C\Big(\sum_{i=1}^{N}\Big\|\sum_{n=1}^{n_0} r_n \pi_i T_n f_n\Big\|^p_{\mathrm{L}^p(0,1;\mathrm{L}^p(\Omega))}\Big)^{\frac{1}{p}}\end{aligned}$$

with a constant $C > 0$ depending only on p and N. Finally, note that again by equivalence of norms in $\mathbb{C}^N$, we find

$$\begin{aligned}\frac{1}{C^p}\sum_{i=1}^{N}\Big(\sum_{n=1}^{n_0} |\pi_i T_n f_n|^2\Big)^{\frac{p}{2}} &\le \Big(\sum_{n=1}^{n_0}\sum_{i=1}^{N} |\pi_i T_n f_n|^2\Big)^{\frac{p}{2}} \\ &\le C^p\sum_{i=1}^{N}\Big(\sum_{n=1}^{n_0} |\pi_i T_n f_n|^2\Big)^{\frac{p}{2}}.\end{aligned}$$

This implies that we can compare

$$\Big\|\sum_{n=1}^{n_0} r_n T_n f_n\Big\|_{\mathrm{L}^p(0,1;X)} \quad \text{and} \quad \Big\|\Big(\sum_{n=1}^{n_0} |T_n f_n|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\Omega)}$$

with an implicit constant depending only on p and N. Since the calculations above remain true for the sum without the T_n's, we conclude the proof. $\square$

A glimpse on the square function estimate (2.4) suggests another formulation of $\mathcal{R}$-boundedness for families of operators on closed subspaces X

of $\mathrm{L}^p(\Omega;\mathbb{C}^N)$. This was also observed by Kunstmann and Weis in [61, p. 89]. For this purpose let

$$X(\ell^2) := \{(f_n)_{n\in\mathbb{N}} \in \mathrm{L}^p(\Omega;\ell^2(\mathbb{C}^N)) : f_n \in X \text{ for all } n \in \mathbb{N}\}$$

endowed with the norm

$$\|f\|_{X(\ell^2)} := \Big\| \Big(\sum_{n=1}^{\infty} |f_n|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\Omega)}.$$

Defining for $n_0 \in \mathbb{N}$ and a finite choice $T_1, \dots, T_{n_0} \in \mathcal{T}$ the operator

$$(T_1, \dots, T_{n_0}, 0, \dots) : X(\ell^2) \to X(\ell^2), \quad f \mapsto (T_1 f_1, \dots, T_{n_0} f_{n_0}, 0, \dots),$$

we arrive at the following proposition.

Proposition 2.3.4. *Let $N \in \mathbb{N}$ and $1 \le p < \infty$. If $X \subset \mathrm{L}^p(\Omega;\mathbb{C}^N)$ is a closed subspace and $\mathcal{T} \subset \mathcal{L}(X)$, then $\mathcal{T}$ is $\mathcal{R}$-bounded if and only if the family of operators*

$$\{(T_1, \dots, T_{n_0}, 0, \dots) : n_0 \in \mathbb{N}, T_1, \dots, T_{n_0} \in \mathcal{T}\}$$

is bounded in $\mathcal{L}(X(\ell^2))$.

The following theorem of Weis [99, Thm. 4.2] builds the bridge between $\mathcal{R}$-boundedness and maximal L^q-regularity.

Theorem 2.3.5 (Weis)**.** *Let $-A$ be the generator of a bounded analytic semigroup on a Banach space X of class $\mathcal{HT}$. Then A has maximal L^q-regularity for one/all $q \in (1,\infty)$ if and only if $\{\lambda(\lambda + A)^{-1} : \lambda \in \mathrm{S}_\theta\}$ is $\mathcal{R}$-bounded for some $\theta > \pi/2$.*

2.4 Embeddings of anisotropic spaces

Suppose that A has maximal L^q-regularity for some $q \in (1,\infty)$ and that additionally $0 \in \rho(A)$. In this case

$$\|x\|_X \le \|A^{-1}\|_{\mathcal{L}(X)} \|Ax\|_X \qquad (x \in X),$$

so that mild solutions of (ACP) lie in the space

$$\mathbb{E} := \mathrm{W}^{1,q}(0,\infty;X) \cap \mathrm{L}^q(0,\infty;\mathcal{D}(A)).$$

If $X = \mathrm{L}^p(\Omega;\mathbb{C}^N)$ and if A is an operator connected to a partial differential equation, the domain of A usually contains functions with some regularity. Hence, it is reasonable to assume that $\mathcal{D}(A)$ embeds into a Bessel potential space $\mathrm{H}^{s,p}(\Omega;\mathbb{C}^N)$ for some $s > 0$. In this case, (and by virtue of Theorem 1.2.4) the intersection above satisfies the continuous embedding

$$\mathbb{E} \subset \mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\Omega;\mathbb{C}^N)).$$

In the treatment of nonlinear problems, it will be eminent to handle these types of spaces and to have embeddings at hand. Please consult Denk and Kaip [19, Lem. 2.61] for a proof of the following result.

Theorem 2.4.1. *Let $1 < p, q < \infty$. Then for every $s \geq 0$ and $\sigma \in [0,1]$ the continuous embedding*

$$\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d)) \subset \mathrm{H}^{\sigma,q}(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\mathbb{R}^d))$$

holds.

Remark 2.4.2. For $p = q$ the theorem above was proven by Denk, Saal, and Seiler [20, Lem. 4.3]. For $p \neq q$ the theorem was proven by Denk and Kaip only for functions that vanish at the origin. One obtains the theorem for general $u \in \mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d))$ by performing the following four steps.

(1) Perform an even reflection in order to extend u to a function $\widetilde{u} \in \mathrm{H}^{1,q}(\mathbb{R};\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(\mathbb{R};\mathrm{H}^{s,p}(\mathbb{R}^d))$.

(2) Multiply this function with a smooth cut-off function φ which is equal to one on $[0,\infty)$ and zero on $(-\infty,-1/2]$.

(3) Shift the function $\widetilde{u}\varphi$ by one to the right. This yields a function in $\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d))$ which vanishes at the origin.

(4) Use the embedding

$$\mathrm{H}_0^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d)) \subset \mathrm{H}_0^{\sigma,q}(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\mathbb{R}^d))$$

proven in [19, Lem. 2.61], where the subscript 0 indicates that the functions vanish at zero.

Combining the embedding above with Sobolev's embedding theorem, see Meyries and Veraar [71, Cor. 1.4] one obtains the following corollary.

Corollary 2.4.3. *Let $1 < p, q < \infty$. Then for every $s \geq 0$, $\sigma \in [0, 1/q)$, and for every $r \in [q, \frac{q}{1-\sigma q}]$ the continuous embedding*

$$\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d)) \subset \mathrm{L}^r(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\mathbb{R}^d))$$

holds. Furthermore, for every $\sigma \in [1/q, 1]$ the continuous embedding

$$\begin{aligned}\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}&(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d;\mathbb{C}^N)) \\ &\subset \mathrm{L}^\infty(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\mathbb{R}^d;\mathbb{C}^N))\end{aligned}$$

holds.

If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then it was shown by Jerison and Kenig [53, Prop. 2.4] that Stein's extension operator gives rise to a bounded linear operator from $\mathrm{H}^{(1-\sigma)s,p}(\Omega)$ into $\mathrm{H}^{(1-\sigma)s,p}(\mathbb{R}^d)$ for every $s \geq 0$ and $\sigma \in [0,1]$. Extending functions in

$$\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\Omega;\mathbb{C}^N))$$

by applying Stein's extension operator to each component of the $\mathbb{C}^N$-valued function yields a function in

$$\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\mathbb{R}^d;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\mathbb{R}^d;\mathbb{C}^N)).$$

Combining the corollary above and the boundedness of Stein's extension operator yields the following result.

Corollary 2.4.4. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and let $1 < p, q < \infty$. Then for every $s \geq 0$, $\sigma \in [0, 1/q)$, and for every $r \in [q, \frac{q}{1-\sigma q}]$ the continuous embedding*

$$\begin{aligned} &\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\Omega;\mathbb{C}^N)) \\ &\qquad\qquad \subset \mathrm{L}^r(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\Omega;\mathbb{C}^N)) \end{aligned}$$

holds. Furthermore, for every $\sigma \in [1/q, 1]$ the continuous embedding

$$\begin{aligned} &\mathrm{H}^{1,q}(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^N)) \cap \mathrm{H}^{0,q}(0,\infty;\mathrm{H}^{s,p}(\Omega;\mathbb{C}^N)) \\ &\qquad\qquad \subset \mathrm{L}^\infty(0,\infty;\mathrm{H}^{(1-\sigma)s,p}(\Omega;\mathbb{C}^N)) \end{aligned}$$

holds.

CHAPTER 3

A Banach space valued version of Shen's L^p-extrapolation theorem

In this chapter, we discuss the L^p-extrapolation theorem of Shen. This theorem provides some conditions, under which one can derive the L^p-boundedness of an operator, which is a priori only L^2-bounded. Such theorems are well-known in harmonic analysis from the classical Calderón–Zygmund theory and are of the following fashion.

Suppose an operator T is a convolution operator with an associated kernel. Suppose that the kernel satisfies further estimates (we will not go into detail here) and that the operator T is bounded on L^2, then T is bounded on L^p for all $1 < p < \infty$. See for example Grafakos [44, Sec. 4.3] and Stein [92, Ch. 2] for respective results.

Now, there are some examples in the theory of elliptic partial differential equations, where certain associated operators are L^p-bounded for some p's in an interval around two, but not for all $1 < p < \infty$. In order to prove the L^p-boundedness of such operators, the classical theorems of singular integrals are worthless, as they are not sensitive for the L^p-boundedness on a proper subinterval of $(1, \infty)$. They provide L^p-boundedness either for all p's or for no p's.

Shen's L^p-extrapolation theorem possesses this sensitivity. It provides a means to conclude that if T is L^2-bounded and if the operator satisfies certain estimates depending on a number $p > 2$, then the operator is L^q-bounded for all $2 < q < p$. This was proven for operators acting on the whole space in [87, Thm. 3.1] and on bounded Lipschitz domains in [87, Thm. 3.3].

3.1 Formulation of the L^p-extrapolation theorem

First of all, we fix some notation. For the rest of this chapter, d is supposed to be a positive integer. Moreover, for a positive number α and a ball $B := B(x_0, r)$, we adopt the notation $\alpha B := B(x_0, \alpha r)$.

The L^p-extrapolation theorem of Shen then reads as follows.

Theorem 3.1.1 (Shen). *Let $\Omega \subset \mathbb{R}^d$ be either the whole space or a bounded Lipschitz domain and let $T \in \mathcal{L}(\mathrm{L}^2(\Omega))$.*

Suppose that there exist constants $p > 2$, $R_0 > 0$, $\alpha_2 > \alpha_1 > 1$, and $\mathcal{C} > 0$, where $R_0 = \infty$ if $\Omega = \mathbb{R}^d$, such that the following holds. For all $B = B(x_0, r)$ with $0 < r < R_0$, which are either centered on $\partial\Omega$, i.e., $x_0 \in \partial\Omega$, or satisfy $\alpha_2 B \subset \Omega$, and all compactly supported $f \in \mathrm{L}^\infty(\Omega)$ with $f = 0$ on $\Omega \cap \alpha_2 B$ the estimate

$$\Big(\frac{1}{r^d}\int_{\Omega\cap B} |Tf|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq \mathcal{C}\bigg\{\Big(\frac{1}{r^d}\int_{\Omega\cap\alpha_1 B} |Tf|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \sup_{B'\supset B}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'} |f|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\}$$

holds. Here, the supremum runs over all balls B' containing B.

Then for each $2 < q < p$ the restriction of T onto $\mathrm{L}^2(\Omega) \cap \mathrm{L}^q(\Omega)$ extends to a bounded linear operator on $\mathrm{L}^q(\Omega)$, with operator norm bounded by a constant depending on d, p, q, α_1, α_2, $\mathcal{C}$, the operator norm of T on L^2, and additionally on R_0, $\mathrm{diam}(\Omega)$, and $|\Omega|$ if Ω is a bounded Lipschitz domain.

To prove the result above, Shen's line of action is to first establish the whole space case and then to imitate the proof for bounded Lipschitz domains. For this imitation, the boundedness of Ω is crucial. The Lipschitz geometry enters the game in order to establish the validity of the estimates in the theorem for *all* balls with non-empty intersection with Ω and radius bounded by R_0. This is done by means of a covering argument.

As the L^p-extrapolation result holds for *bounded* domains and one very special *unbounded* domain (on $\mathbb{R}^d$), a natural question to ask is whether the theorem remains true for other types of unbounded domains. Furthermore, spending more effort on the covering argument mentioned above, by appealing for example to Vitali's covering lemma, which does not require any geometry at all, one can hope to relax the geometrical setup of the extrapolation theorem. Finally, it was already observed by Auscher in the remarks below [6, Thm. 1.2], that Shen's L^p-extrapolation theorem holds in the Banach space valued setting as well.

We will follow a different strategy than Shen by first establishing the whole space result in the Banach space valued setting, thereby verifying Auscher's observation, and then we will reduce the extrapolation theorem for any other type of domain to the whole space case. The result we prove reads as follows.

Theorem 3.1.2. *Let X and Y be Banach spaces, $\Omega \subset \mathbb{R}^d$ be open, $\mathcal{M} > 0$, and let $T \in \mathcal{L}(\mathrm{L}^2(\Omega; X), \mathrm{L}^2(\Omega; Y))$ with $\|T\|_{\mathcal{L}(\mathrm{L}^2(\Omega;X),\mathrm{L}^2(\Omega;Y))} \leq \mathcal{M}$.*

Suppose that there exist constants $p > 2$, $R_0 > 0$, $\alpha_2 > \alpha_1 > 1$, and $\mathcal{C} > 0$, where $R_0 = \infty$ if $\operatorname{diam}(\Omega) = \infty$, such that the following holds. For all $B = B(x_0, r)$ with $0 < r < R_0$, which are either centered on $\partial\Omega$, i.e., $x_0 \in \partial\Omega$, or satisfy $\alpha_2 B \subset \Omega$, and all compactly supported $f \in \mathrm{L}^\infty(\Omega; X)$ with $f = 0$ on $\Omega \cap \alpha_2 B$ the estimate

$$
\begin{aligned}
\Big(\frac{1}{r^d}\int_{\Omega\cap B} \|Tf\|_Y^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq \mathcal{C}\Big\{ &\Big(\frac{1}{r^d}\int_{\Omega\cap\alpha_1 B} \|Tf\|_Y^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&+ \sup_{B'\supset B}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'} \|f\|_X^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}\Big\}
\end{aligned}
\tag{3.1}
$$

holds. Here the supremum runs over all balls B' containing B.

Then for each $2 < q < p$ the restriction of T onto $\mathrm{L}^2(\Omega; X) \cap \mathrm{L}^q(\Omega; X)$ extends to a bounded linear operator from $\mathrm{L}^q(\Omega; X)$ into $\mathrm{L}^q(\Omega; Y)$, with operator norm bounded by a constant depending on d, p, q, α_1, α_2, $\mathcal{C}$, and $\mathcal{M}$, and additionally on R_0 and $\operatorname{diam}(\Omega)$ if Ω is bounded.

Remark 3.1.3. (1) If $\mathcal{T} \subset \mathcal{L}(\mathrm{L}^2(\Omega; X), \mathrm{L}^2(\Omega; Y))$ is a bounded family of operators one immediately sees that if one can verify the assumptions of Theorem 3.1.2 with uniform constants for every operator in $\mathcal{T}$, then each operator extends to $\mathrm{L}^q(\Omega; X)$, yielding a bounded family of operators in $\mathcal{L}(\mathrm{L}^q(\Omega; X), \mathrm{L}^q(\Omega; Y))$.

(2) In this treatise, we will verify that (3.1) holds true without the second term on the right-hand side in each particular situation. Such an estimate will be called a *weak reverse Hölder estimate* and we say that an operator T satisfies weak reverse Hölder estimates if it satisfies the assumptions of Theorem 3.1.2 without the second term on the right-hand side of (3.1).

Weak reverse Hölder estimates satisfy a self-improving property. This is due to Giaquinta and Modica [38, Prop. 5.1] and can be found in the formulation below in Giaquinta and Martinazzi [37, Thm. 6.38]. Note that the explicit dependence of the constants is formulated in [38, Prop. 5.1].

Proposition 3.1.4 (Giaquinta, Modica). *Let $\Omega \subset \mathbb{R}^d$ be open, $f \in \mathrm{L}^q_{\mathrm{loc}}(\Omega)$, $q > 1$, be a non-negative function. Suppose for some constants $b > 0$, $R_0 > 0$*

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} f^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq \frac{b}{r^d} \int_{B(x_0,2r)} f \, \mathrm{d}x$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0, \partial\Omega)/2\}$. Then $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$, depending only on d, q, and b, and there is a constant C depending only on d, q, ε, and b such that

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} f^{q+\varepsilon} \, \mathrm{d}x \right)^{\frac{1}{q+\varepsilon}} \leq C \left(\frac{1}{r^d} \int_{B(x_0,2r)} f^q \, \mathrm{d}x \right)^{\frac{1}{q}}$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0, \partial\Omega)/2\}$.

3.2 Some preliminaries

For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ define the *maximal operator* as

$$(\mathrm{M}f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y \qquad (x \in \mathbb{R}^d)$$

and for a set $\mathcal{Q}$ being a ball or a cube, and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{Q})$ define the *localized maximal operator* via

$$(3.2)\qquad (\mathrm{M}_{\mathcal{Q}} f)(x) := \sup_{\substack{Q \ni x \\ Q \subset \mathcal{Q}}} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y \qquad (x \in \mathcal{Q}),$$

where in the suprema Q denotes a cube containing x. The localized maximal operator shares the same properties (as an operator on $\mathrm{L}^p(\mathcal{Q})$) as the maximal operator on $\mathbb{R}^d$ and these are listed in the following proposition. The proofs of these properties follow for the localized maximal operator literally the same lines as for the usual maximal operator, see Stein [92, Thm. I.1].

Proposition 3.2.1. *Let $\mathcal{Q} \subset \mathbb{R}^d$ be a ball or a cube. Then there exists a constant $C_d > 0$, solely depending on d, such that*

$$\begin{aligned} \|\mathrm{M} f\|_{\mathrm{L}^{1,\infty}(\mathbb{R}^d)} &\leq C_d \|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \qquad (f \in \mathrm{L}^1(\mathbb{R}^d)) \\ \|\mathrm{M}_{\mathcal{Q}} f\|_{\mathrm{L}^{1,\infty}(\mathcal{Q})} &\leq C_d \|f\|_{\mathrm{L}^1(\mathcal{Q})} \qquad (f \in \mathrm{L}^1(\mathcal{Q})) \end{aligned}$$

holds. Moreover, if $1 < p \leq \infty$ there exists a constant $C_{p,d} > 0$, depending on d and p, such that

$$\begin{aligned} \|\mathrm{M} f\|_{\mathrm{L}^p(\mathbb{R}^d)} &\leq C_{p,d} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \qquad (f \in \mathrm{L}^p(\mathbb{R}^d)) \\ \|\mathrm{M}_{\mathcal{Q}} f\|_{\mathrm{L}^p(\mathcal{Q})} &\leq C_{p,d} \|f\|_{\mathrm{L}^p(\mathcal{Q})} \qquad (f \in \mathrm{L}^p(\mathcal{Q})). \end{aligned}$$

The following lemma contains the covering argument mentioned in the introduction of this chapter.

Lemma 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ be open, $f, g \in \mathrm{L}^2(\Omega)$, $\alpha_2 > \alpha_1 > 1$, $p > 2$, and $r > 0$ and $x_0 \in \mathbb{R}^d$ be such that $B(x_0, r) \cap \Omega \neq \emptyset$. If there exists $C > 0$ such that*

$$\begin{aligned} \Big(\frac{1}{r^d} \int_{\Omega \cap \widetilde{B}} |f|^p \, \mathrm{d}x \Big)^{\frac{1}{p}} \leq C \bigg\{ &\Big(\frac{1}{r^d} \int_{\Omega \cap \alpha_1 \widetilde{B}} |f|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\ &+ \sup_{B' \supset \widetilde{B}} \Big(\frac{1}{|B'|} \int_{\Omega \cap B'} |g|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \bigg\} \end{aligned}$$

holds for all balls $\widetilde{B}$ with $\alpha_2 \widetilde{B} \subset B(x_0, \alpha_2 r)$ and which are either centered on $\partial\Omega$ or satisfy $\alpha_2 \widetilde{B} \subset \Omega$, then, for each $\alpha \in (1, \alpha_2)$ there exists a constant C' such that

$$\Big(\frac{1}{r^d}\int_{\Omega \cap B(x_0,r)} |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq C' \Big\{ \Big(\frac{1}{r^d}\int_{\Omega \cap B(x_0,\alpha r)} |f|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \sup_{B' \supset B(x_0,r)} \Big(\frac{1}{|B'|}\int_{\Omega \cap B'} |g|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big\},$$

where C' depends on d, α, α_1, α_2, p, and C.

Proof. Define

$$c := \min\Big\{\frac{\alpha_2 - 1}{5\alpha_2 + 1}, \frac{\alpha - 1}{5\alpha_1 + 1}\Big\}$$

and

$$\begin{aligned} \mathcal{I}_1 &:= \{y \in \Omega \cap B(x_0, r) : B(y, cr) \subset \Omega\}, \\ \mathcal{I}_2 &:= \{y' \in \partial\Omega : \text{there is } y \in \Omega \cap B(x_0, r) \text{ such that } y' \in B(y, cr)\}. \end{aligned}$$

Note that for all $y \in \mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2$ we have $B(y, 5c\alpha_1 r) \subset B(x_0, \alpha r)$ and $B(y, 5c\alpha_2 r) \subset B(x_0, \alpha_2 r)$ by definition of c. Moreover, by construction

$$\Omega \cap B(x_0, r) \subset \Omega \cap \bigcup_{y \in \mathcal{I}} B(y, cr).$$

The covering lemma of Vitali, see Evans and Gariepy [27, Thm. 1.5.1], yields an at most countable index set $\mathcal{F}$ such that all balls in the family $\{B(y, cr)\}_{y \in \mathcal{F}}$ are pairwise disjoint and such that

$$\bigcup_{y \in \mathcal{I}} B(y, cr) \subset \bigcup_{y \in \mathcal{F}} B(y, 5cr).$$

Furthermore, for $\sharp(\mathcal{F})$ being the number of points in $\mathcal{F}$, we have

$$|B(0,1)|\, c^d r^d \sharp(\mathcal{F}) = \sum_{y \in \mathcal{F}} |B(y, cr)| \leq |B(x_0, \alpha r)| = |B(0,1)|\, \alpha^d r^d,$$

that is $\sharp(\mathcal{F}) \leq (\alpha/c)^d$. This yields by hypothesis

$$\begin{aligned}\frac{1}{r^d}\int_{\Omega\cap B(x_0,r)} |f|^p \,\mathrm{d}x &\leq \frac{1}{r^d}\sum_{y\in\mathcal{F}}\int_{\Omega\cap B(y,5cr)} |f|^p \,\mathrm{d}x \\ &\leq C^p \sum_{y\in\mathcal{F}}\bigg\{\bigg(\frac{1}{r^d}\int_{\Omega\cap B(y,5\alpha_1 cr)} |f|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\qquad + \sup_{B'\supset B(y,5cr)}\bigg(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}^p.\end{aligned}$$

Define $\beta := (2+c)/(5c)$ and note that $\beta \geq 1$. Using this together with $B(y, 5c\alpha_1 r) \subset B(x_0, \alpha r)$ delivers

$$\begin{aligned}&\leq \beta^{\frac{dp}{2}} C^p \sum_{y\in\mathcal{F}}\bigg\{\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha r)} |f|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\qquad + \sup_{\beta B'\supset B(y,5c\beta r)}\bigg(\frac{1}{|\beta B'|}\int_{\Omega\cap\beta B'} |g|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}^p.\end{aligned}$$

Finally, the choice of β ensures $B(x_0, r) \subset B(y, 5c\beta r)$ for all $y \in \mathcal{F}$. Thus, the supremum becomes larger if we replace $\beta B'$ by arbitrary balls that contain $B(x_0, r)$. This implies

$$\begin{aligned}&\leq \beta^{\frac{dp}{2}} C^p \sharp(\mathcal{F})\bigg\{\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha r)} |f|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\qquad + \sup_{B'\supset B(x_0,r)}\bigg(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}^p\end{aligned}$$

and concludes the proof. □

The following lemma is proven similarly as the previous one and shows that the actual size of the number R_0 in Theorem 3.1.2 is of no great importance.

Lemma 3.2.3. *Let $R_0' > R_0 > 0$, $f, g \in \mathrm{L}^2(\mathbb{R}^d)$, $\alpha > 1$, and $p > 2$. If there exists a constant $C > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $0 < r < R_0$*

the inequality

$$\Big(\frac{1}{r^d}\int_{B(x_0,r)} |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq C\bigg\{\Big(\frac{1}{r^d}\int_{B(x_0,\alpha r)} |f|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \sup_{B'\supset B(x_0,r)}\Big(\frac{1}{|B'|}\int_{B'} |g|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\}$$

holds. Then there exists a constant C' depending on C, α, R_0, R_0', and d such that the same inequality holds for all $0 < r < R_0'$ with C replaced by C'.

Proof. Let $x_0 \in \mathbb{R}^d$, $r \in [R_0, R_0')$, and $B := B(x_0, r)$. Define $\beta := \min\{(\alpha-1)/(5\alpha), R_0/(5R_0')\}$. It is clear that $\{B(y,\beta r)\}_{y\in B}$ is an open covering of B. The covering lemma of Vitali [27, Thm. 1.5.1] yields an at most countable subset $\mathcal{F} \subset B$ such that the balls $\{B(y,\beta r)\}_{y\in\mathcal{F}}$ are pairwise disjoint and such that

$$B \subset \bigcup_{y\in\mathcal{F}} B(y, 5\beta r).$$

Furthermore, since $\alpha > 1$ we find $\beta \leq \alpha - 1$ and conclude that for each $y \in \mathcal{F}$ we have $B(y,\beta r) \subset \alpha B$. Consequently, we can estimate

$$|B(0,1)|\,\beta^d r^d \sharp(\mathcal{F}) = \sum_{y\in\mathcal{F}} |B(y,\beta r)| \leq |B(0,1)|\,\alpha^d r^d$$

and find that $\sharp(\mathcal{F}) \leq (\alpha/\beta)^d$. Now,

$$\Big(\frac{1}{r^d}\int_B |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq [5\beta]^{\frac{d}{p}} \sum_{y\in\mathcal{F}} \Big(\frac{1}{[5\beta r]^d}\int_{B(y,5\beta r)} |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}$$

and since $5\beta r < R_0$

$$\leq [5\beta]^{\frac{d}{p}} C \sum_{y\in\mathcal{F}} \bigg\{\Big(\frac{1}{[5\beta r]^d}\int_{B(y,5\alpha\beta r)} |f|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \sup_{B'\supset B(y,5\beta r)}\Big(\frac{1}{|B'|}\int_{B'} |g|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\}.$$

Next, $5\alpha\beta \leq \alpha - 1$ and $B(y,(\alpha-1)r) \subset \alpha B$ imply that the first integral on the right-hand side is controlled by

$$\left(\frac{1}{[5\beta r]^d}\int_{B(y,5\alpha\beta r)} |f|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{1}{[5\beta]^{\frac{d}{2}}}\left(\frac{1}{r^d}\int_{\alpha B} |f|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}.$$

For the supremum, we first use that $5\beta \leq 2$ and then, that the arising averages are taken solely on balls that contain $B(y,2r)$. Since $B \subset B(y,2r)$ for every $y \in \mathcal{F}$, the supremum will be larger if it runs over all balls that contain B. Indeed,

$$\begin{aligned}
&\sup_{B' \supset B(y,5\beta r)} \left(\frac{1}{|B'|}\int_{B'} |g|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\frac{2}{5\beta}\right)^{\frac{d}{2}} \sup_{B' \supset B(y,5\beta r)} \left(\frac{1}{|\frac{2}{5\beta}B'|}\int_{\frac{2}{5\beta}B'} |g|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\frac{2}{5\beta}\right)^{\frac{d}{2}} \sup_{B' \supset B} \left(\frac{1}{|B'|}\int_{B'} |g|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

We conclude the proof by recalling the bound on $\sharp(\mathcal{F})$. □

The proof of Theorem 3.1.2 requires a version of the Calderón–Zygmund decomposition proven by Caffarelli and Peral [14, Lem. 1.1].

Lemma 3.2.4 (Caffarelli, Calderón, Peral, Zygmund)**.** *Let Q be a bounded cube in $\mathbb{R}^d$ and $\mathcal{A} \subset Q$ a measurable set satisfying*

$$0 < |\mathcal{A}| < \delta|Q| \qquad \textit{for some } 0 < \delta < 1.$$

Then there is a family of disjoint dyadic cubes $\{Q_k\}_{k\in\mathbb{N}}$ obtained from Q by bisecting the sides of Q, such that for all $k \in \mathbb{N}$

a) $|\mathcal{A} \setminus \bigcup_{l\in\mathbb{N}} Q_l| = 0$, *b)* $|\mathcal{A} \cap Q_k| > \delta|Q_k|$, *c)* $|\mathcal{A} \cap Q_k^*| \leq \delta|Q_k^*|$,

where Q_k^* is the dyadic parent of Q_k.

3.3 The proof of Theorem 3.1.2

In order to prove Theorem 3.1.2, we proceed in several steps. We start by establishing the Banach space valued whole space case by imitating the proof of the scalar-valued case in [87, Thm. 3.1]. Then, for $\Omega \neq \mathbb{R}^d$, we consider the auxiliary operator

$$T_{\mathbb{R}^d} : \mathrm{L}^2(\mathbb{R}^d; X) \to \mathrm{L}^2(\mathbb{R}^d; Y), \quad f \mapsto \chi_\Omega T \chi_\Omega f.$$

One directly verifies that the restriction of $T_{\mathbb{R}^d}$ onto $\mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^q(\mathbb{R}^d; X)$ extends to a bounded operator from $\mathrm{L}^q(\mathbb{R}^d; X)$ into $\mathrm{L}^q(\mathbb{R}^d; Y)$ if and only if this is the case for T (with regard to the respective spaces on Ω). We aim to prove that $T_{\mathbb{R}^d}$ satisfies the premises of the whole spaces result supposed that T satisfies those on Ω. However, if Ω is not the whole space, we assumed the validity of the estimates in (3.1) only for balls that are either centered on the boundary or whose dilation by the factor α_2 is completely contained inside Ω. In comparison to that, the whole space result requires the validity of the estimates for *all* balls in $\mathbb{R}^d$. Lemma 3.2.2 will take care of the weak reverse Hölder estimates for balls which neither are centered on the boundary of Ω nor lie completely inside Ω. Finally, if Ω is bounded, we will see that $T_{\mathbb{R}^d} f$ satisfies the weak reverse Hölder estimates trivially for radii which are chosen large enough. The bridge between this "large enough" and the number R_0 from the presumptions of the theorem is built by an application of Lemma 3.2.3.

Proof of Theorem 3.1.2. For this proof, we denote a generic constant that depends solely on d, p, q, R_0, α_1, α_2, or the constant $\mathcal{C}$ by C_g. We will abbreviate the operator norm $\|T\|_{\mathcal{L}(\mathrm{L}^2(\Omega;X),\mathrm{L}^2(\Omega;Y))}$ by $\|T\|$.
Step 1: Verification of the case $\Omega = \mathbb{R}^d$.

Let $x_0 \in \mathbb{R}^d$, $r > 0$, and Q be a cube in $\mathbb{R}^d$ with $\operatorname{diam}(Q) = 2r$ and midpoint x_0. Let $B := B(x_0, r)$. One directly verifies that

$$\frac{1}{\sqrt{d}} B \subset Q \subset B \quad \text{and} \quad |Q| = \Big(\frac{2r}{\sqrt{d}}\Big)^d.$$

Define $\alpha_1' := \sqrt{d}\alpha_1$ and $\alpha_2' := \sqrt{d}\alpha_2$ so that by means of the weak reverse Hölder estimates for all compactly supported $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ with $f = 0$

on $\alpha_2' Q$ the inequality

$$
\begin{aligned}
\Big(\frac{1}{|Q|} \int_Q \|Tf\|_Y^p \, \mathrm{d}x \Big)^{\frac{1}{p}} &\leq C_g \Big(\frac{1}{r^d} \int_B \|Tf\|_Y^p \, \mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq C_g \bigg\{ \Big(\frac{1}{r^d} \int_{\alpha_1 B} \|Tf\|_Y^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\
&\qquad + \sup_{B' \supset B} \Big(\frac{1}{|B'|} \int_{B'} \|f\|_X^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \bigg\} \\
&\leq C_g \bigg\{ \Big(\frac{1}{|\alpha_1' Q|} \int_{\alpha_1' Q} \|Tf\|_Y^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\
&\qquad + \sup_{Q' \supset Q} \Big(\frac{1}{|Q'|} \int_{Q'} \|f\|_X^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \bigg\}
\end{aligned}
\tag{3.3}
$$

holds. Here, the last supremum runs over all cubes Q' containing Q and βQ denotes the by $\beta > 0$ dilated cube with same center as Q. Thus, without loss of generality, we can replace balls by cubes in the assumption of the theorem.

Fix $q \in (2, p)$ and take $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ with compact support. For $\lambda > 0$ consider the set

$$
E(\lambda) := \{ x \in \mathbb{R}^d : \mathrm{M}(\|Tf\|_Y^2)(x) > \lambda \}.
$$

Since $\|Tf\|_Y^2 \in \mathrm{L}^1(\mathbb{R}^d)$, the weak-type estimate of the maximal operator, see Proposition 3.2.1, implies

$$
|E(\lambda)| \leq \frac{C_g}{\lambda} \Big\| \|Tf\|_Y^2 \Big\|_{\mathrm{L}^1(\mathbb{R}^d)} = \frac{C_g}{\lambda} \|Tf\|_{\mathrm{L}^2(\mathbb{R}^d; Y)}^2. \tag{3.4}
$$

Let $A := 1/(2\delta^{2/q}) > 5^d$, where $\delta \in (0,1)$ is a small constant to be determined. Decompose $\mathbb{R}^d$ into a dyadic grid. Then by (3.4) we find a mesh size such that each cube Q_0 from the grid satisfies

$$
|E(A\lambda)| < \delta \, |Q_0| \, .
$$

Note that the mesh size is allowed to depend on λ, δ, f, and T. If the case $|Q_0 \cap E(A\lambda)| = 0$ applies, do nothing. In the other case, the set defined by $\mathcal{A} := Q_0 \cap E(A\lambda)$ together with the cube Q_0 satisfy the assumptions of Lemma 3.2.4. Proceeding in that way for every cube Q_0 in the grid and enumerating all cubes obtained in this way by Lemma 3.2.4 by $\{Q_k\}_{k\in\mathbb{N}}$ yields a countable family of mutually disjoint cubes satisfying for all $k \in \mathbb{N}$

(i) $|E(A\lambda) \setminus \bigcup_{l\in\mathbb{N}} Q_l| = 0$;

(ii) $|E(A\lambda) \cap Q_k| > \delta|Q_k|$;

(iii) $|E(A\lambda) \cap Q_k^*| \leq \delta|Q_k^*|$.

Note that as in Lemma 3.2.4, Q_k^* denotes the dyadic parent of Q_k.
Claim 1: The operator T is L^q-bounded, once there are constants $\delta, \gamma > 0$ such that for all $\lambda > 0$

$$|E(A\lambda)| \leq \delta|E(\lambda)| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2)(x) > \lambda\gamma\}| \tag{3.5}$$

holds.

To see this, first note that (3.4) and $q > 2$ imply that the function $\lambda \mapsto \lambda^{q/2-1}|E(\lambda)|$ is in $\mathrm{L}^1_{\mathrm{loc}}([0,\infty))$. The premise of Claim 1 implies that for all $\lambda_0 > 0$

$$\begin{aligned}\int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)| \,\mathrm{d}\lambda \leq \delta \int_0^{A\lambda_0} &\lambda^{\frac{q}{2}-1}|E(A^{-1}\lambda)| \,\mathrm{d}\lambda \\ &+ \int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2)(x) > A^{-1}\lambda\gamma\}| \,\mathrm{d}\lambda.\end{aligned}$$

Next, use linear transformation and the definition of A in the first integral, and estimate the second integral by sending λ_0 to infinity, to get

$$\begin{aligned}&\leq 2^{-\frac{q}{2}} \int_0^{\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)| \,\mathrm{d}\lambda \\ &\quad+ \int_0^{\infty} \lambda^{\frac{q}{2}-1}|\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2)(x) > A^{-1}\lambda\gamma\}| \,\mathrm{d}\lambda.\end{aligned}$$

Appeal to linear transformation and Proposition 1.1.14 to conclude that the second integral coincides modulo a factor by a generic constant times $\gamma^{-q/2}$ with $\|\mathrm{M}(\|f\|_X^2)\|^{q/2}_{\mathrm{L}^{q/2}(\mathbb{R}^d)}$. By virtue of the boundedness of the maximal operator on $\mathrm{L}^{q/2}$, cf. Proposition 3.2.1, this results in the estimate

$$\leq 2^{-\frac{q}{2}} \int_0^{\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)| \,\mathrm{d}\lambda + \delta\gamma^{-\frac{q}{2}} C_g \|f\|^q_{\mathrm{L}^q(\mathbb{R}^d;X)}.$$

Using $A > 1$ and $q > 2$, the left-hand side of the inequality can absorb the first term on the right-hand side. This yields

$$\int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1} |E(\lambda)| \, \mathrm{d}\lambda \leq \delta \gamma^{-\frac{q}{2}} C_g \|f\|^q_{\mathrm{L}^q(\mathbb{R}^d;X)}.$$

Taking $\lambda_0 \to \infty$ and using $\|[Tf](x)\|^2_Y \leq \mathrm{M}(\|Tf\|^2_Y)(x)$ for almost every $x \in \mathbb{R}^d$ yields together with Proposition 1.1.14

$$\|Tf\|^q_{\mathrm{L}^q(\mathbb{R}^d;Y)} \leq \gamma^{-\frac{q}{2}} C_g \|f\|^q_{\mathrm{L}^q(\mathbb{R}^d;X)}.$$

The conclusion of the theorem follows by density (note that simple functions with bounded support are dense in all $\mathrm{L}^q(\Omega; X)$-spaces by construction of the Bochner integral).

Claim 2: The premise of Claim 1 follows if there are constants $\delta, \gamma > 0$ such that for all dyadic parents Q_k^* of the family of cubes $\{Q_k\}_{k\in\mathbb{N}}$ constructed before (i)–(iii) the following statement is valid:

$$Q_k^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) \leq \lambda\gamma\} \neq \emptyset \quad \text{implies} \quad Q_k^* \subset E(\lambda).$$

Since (3.5) is trivial if $|E(A\lambda)| = 0$ assume that $|E(A\lambda)| > 0$. Let $I \subset \mathbb{N}$ be the index set of all $l \in \mathbb{N}$ such that $\{Q_l^*\}_{l\in I}$ is a maximal set of mutually disjoint cubes satisfying $Q_l^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) \leq \lambda\gamma\} \neq \emptyset$. Then,

$$\begin{aligned} |E(A\lambda)| = |E(A\lambda) &\cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) \leq \lambda\gamma\}| \\ &+ |E(A\lambda) \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) > \lambda\gamma\}|. \end{aligned}$$

Dealing the first term on the right-hand side by the maximality of $\{Q_l^*\}_{l\in I}$ together with (i) and the second term by using the monotonicity of the Lebesgue measure yields

$$\leq \sum_{l\in I} |E(A\lambda) \cap Q_l^*| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) > \lambda\gamma\}|.$$

Next, use (iii) first and then the mutual disjointness of the family $\{Q_l^*\}_{l\in I}$ together with the assertion of Claim 2 to get

$$\leq \delta |E(\lambda)| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|^2_X)(x) > \lambda\gamma\}|.$$

Claim 3: There exist $\delta, \gamma > 0$ such that

$$Q_k^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2)(x) \leq \lambda\gamma\} \neq \emptyset \quad \text{implies} \quad Q_k^* \subset E(\lambda).$$

To conclude this statement, we argue by contradiction. For this purpose, suppose that there exists a Q_k with $\{x \in Q_k^* : \mathrm{M}(\|f\|_X^2)(x) \leq \gamma\lambda\} \neq \emptyset$ and $Q_k^* \setminus E(\lambda) \neq \emptyset$. We show, that the existence of such a cube contradicts (ii). In this situation, for every cube Q that contains Q_k^*, we have

$$\frac{1}{|Q|} \int_Q \|f\|_X^2 \, \mathrm{d}x \leq \gamma\lambda \qquad \text{and} \qquad \frac{1}{|Q|} \int_Q \|Tf\|_Y^2 \, \mathrm{d}x \leq \lambda. \tag{3.6}$$

Next, let $x \in Q_k$ and Q' be a cube with $x \in Q'$ and $Q' \not\subset 2Q_k^*$. Then, we find for the sidelength of Q' that $\ell(Q') > \ell(Q_k)$. If $y \in Q_k^*$, $1 \leq i \leq d$, and if x' denotes the center of Q', then

$$|y_i - x_i'| \leq |y_i - x_i| + |x_i - x_i'| \leq 2\ell(Q_k) + \frac{1}{2}\ell(Q') < \frac{5}{2}\ell(Q').$$

Consequently, we have $Q_k^* \subset 5Q'$ and thus for $x \in Q_k$

$$\begin{aligned} \mathrm{M}(\|Tf\|_Y^2)(x) &= \max\Big\{\mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x), \sup_{\substack{Q' \ni x \\ Q' \not\subset 2Q_k^*}} \frac{1}{|Q'|} \int_{Q'} \|Tf\|_Y^2 \, \mathrm{d}y\Big\} \\ &\leq \max\{\mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x), 5^d\lambda\}, \end{aligned}$$

where $\mathrm{M}_{2Q_k^*}$ is the localized maximal operator, cf. (3.2). Since $A = 1/(2\delta^{2/q}) > 5^d$, we derive

$$|E(A\lambda) \cap Q_k| \leq |\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x) > A\lambda\}|.$$

Use $(a+b)^2 \leq 2(a^2 + b^2)$ together with Proposition 1.1.13 (1) and (3) to estimate

$$\begin{aligned} &\leq \Big|\Big\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|T(f\chi_{2\alpha_2 Q_k^*})\|_Y^2)(x) > \frac{A\lambda}{4}\Big\}\Big| \\ &\quad + \Big|\Big\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|T(f\chi_{\mathbb{R}^d \setminus 2\alpha_2 Q_k^*})\|_Y^2)(x) > \frac{A\lambda}{4}\Big\}\Big| \\ &=: \mathcal{A} + \mathcal{B}. \end{aligned}$$

By means of the weak-type estimate of $\mathrm{M}_{2Q_k^*}$, see Proposition 3.2.1, in the first inequality below, and the boundedness of T from $\mathrm{L}^2(\mathbb{R}^d;X)$ into $\mathrm{L}^2(\mathbb{R}^d;Y)$ together with (3.6) in the second inequality below, we derive

$$\mathcal{A} \leq \frac{C_g}{A\lambda}\int_{2Q_k^*} \|T(f\chi_{2\alpha_2 Q_k^*})\|_Y^2 \,\mathrm{d}x \leq |Q_k|\|T\|^2\frac{C_g\gamma}{A}.$$

Next, the continuous embedding $\mathrm{L}^{p/2}(2Q_k^*) \subset \mathrm{L}^{p/2,\infty}(2Q_k^*)$, see Proposition 1.1.16, and the $\mathrm{L}^{p/2}$-boundedness of $\mathrm{M}_{2Q_k^*}$, see Proposition 3.2.1, yield

$$\begin{aligned}(A\lambda)^{\frac{p}{2}}\mathcal{B} &\leq C_g\Big\|\mathrm{M}_{2Q_k^*}(\|T(f\chi_{\mathbb{R}^d\setminus 2\alpha_2Q_k^*})\|_Y^2)\Big\|_{\mathrm{L}^{p/2}(2Q_k^*)}^{\frac{p}{2}} \\ &\leq C_g\int_{2Q_k^*}\|T(f\chi_{\mathbb{R}^d\setminus 2\alpha_2Q_k^*})\|_Y^p \,\mathrm{d}x.\end{aligned}$$

An application of (3.3) yields (recall that without loss of generality we changed the values of α_1 and α_2 to α_1' and α_2')

$$\begin{aligned}&\leq |Q_k|C_g\bigg\{\bigg(\frac{1}{|2\alpha_1Q_k^*|}\int_{2\alpha_1Q_k^*}\|T(f\chi_{\mathbb{R}^d\setminus 2\alpha_2Q_k^*})\|_Y^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\qquad + \sup_{Q'\supset 2Q_k^*}\bigg(\frac{1}{|Q'|}\int_{Q'}\|f\chi_{\mathbb{R}^d\setminus 2\alpha_2Q_k^*}\|_X^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}^p.\end{aligned}$$

Add and subtract $f\chi_{2\alpha_2Q_k^*}$ in the argument of T in the first integral and use $\|f\chi_{\mathbb{R}^d\setminus 2\alpha_2Q_k^*}\|_X \leq \|f\|_X$ together with (3.6) in the second integral to obtain

$$\begin{aligned}&\leq |Q_k|C_g\bigg\{\bigg(\frac{1}{|2\alpha_1Q_k^*|}\int_{2\alpha_1Q_k^*}\|Tf\|_Y^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\qquad + \bigg(\frac{1}{|2\alpha_1Q_k^*|}\int_{2\alpha_1Q_k^*}\|T(f\chi_{2\alpha_2Q_k^*})\|_Y^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} + (\gamma\lambda)^{\frac{1}{2}}\bigg\}^p.\end{aligned}$$

Another application of (3.6) to the first integral and a combination of the L^2-boundedness of T with (3.6) in the second integral finally yield

$$\leq |Q_k|C_g\lambda^{\frac{p}{2}}[1+\gamma^{\frac{1}{2}}(1+\|T\|)]^p.$$

Recall that $A = 1/(2\delta^{2/q})$ and that $\|T\| \leq \mathcal{M}$, so that we obtain

$$\begin{aligned}|E(A\lambda) \cap Q_k| &\leq \mathcal{A} + \mathcal{B} \\ &\leq C_g \delta |Q_k| \Big\{ \frac{\gamma \mathcal{M}^2}{A\delta} + \frac{[1 + \gamma^{\frac{1}{2}}(1+\mathcal{M})]^p}{A^{\frac{p}{2}}\delta} \Big\} \\ &\leq C_g \delta |Q_k| \Big\{ \gamma \delta^{\frac{2}{q}-1} \mathcal{M}^2 + [1 + \gamma^{\frac{1}{2}}(1+\mathcal{M})]^p \delta^{\frac{p}{q}-1} \Big\}.\end{aligned}$$

Since $p > q$, we can choose δ such that $[2 + \mathcal{M}]^p \delta^{\frac{p}{q}-1} \leq 1/(2C_g)$. For this fixed value of δ choose $\gamma \leq \min\{1, \delta^{1-\frac{2}{q}}/(2\mathcal{M}^2 C_g)\}$. Then, we obtain $|E(A\lambda) \cap Q_k| \leq \delta |Q_k|$ which is a contradiction to (ii) of the Calderón–Zygmund decomposition.

Step 2: In the case $\Omega \neq \mathbb{R}^d$, we define the auxiliary operator $T_{\mathbb{R}^d}$ and verify the weak reverse Hölder estimates on $B(x_0, r)$ either if $0 < r < R_0$ and $B(x_0, r) \cap \Omega \neq \emptyset$, or $r > 0$ and $B(x_0, r) \cap \Omega = \emptyset$. Furthermore, we conclude the proof for unbounded domains (i.e., $R_0 = \infty$).

For this purpose, define the operator

$$T_{\mathbb{R}^d} : \mathrm{L}^2(\mathbb{R}^d; X) \to \mathrm{L}^2(\mathbb{R}^d; Y), \quad f \mapsto \chi_\Omega T \chi_\Omega f.$$

This operator is well-defined, for if $f \in \mathrm{L}^2(\mathbb{R}^d; X)$ then $\chi_\Omega f \in \mathrm{L}^2(\Omega; X)$. Moreover, note that the restriction of $T_{\mathbb{R}^d}$ onto $\mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^q(\mathbb{R}^d; X)$ extends to a bounded operator from $\mathrm{L}^q(\mathbb{R}^d; X)$ into $\mathrm{L}^q(\mathbb{R}^d; Y)$ if and only if this holds true for T (for the respective spaces on Ω). Using Lemma 3.2.2 we conclude that $T_{\mathbb{R}^d} f$ satisfies weak reverse Hölder estimates on all balls B with $B \cap \Omega \neq \emptyset$ which have a radius less than R_0, whenever f is compactly supported and $f = 0$ on $\alpha_2 B$. Finally, if $B \cap \Omega = \emptyset$, then

$$\int_B \|T_{\mathbb{R}^d} f\|_Y^p \, \mathrm{d}x = 0$$

for all $f \in \mathrm{L}^2(\mathbb{R}^d; Y)$ by definition of the operator $T_{\mathbb{R}^d}$. Consequently, $T_{\mathbb{R}^d}$ satisfies weak reverse Hölder estimates trivially for all balls with $B \cap \Omega = \emptyset$. Thus, in the case $R_0 = \infty$, we can conclude the proof by means of Step 1.

Step 3: If Ω is bounded, we verify that $T_{\mathbb{R}^d}$ satisfies weak reverse Hölder estimates for arbitrary radii and conclude the proof in this case.

By virtue of Step 2 we can restrict ourselves to the situation where $B(x_0, r) \cap \Omega \neq \emptyset$. If $r \geq \operatorname{diam}(\Omega)/(\alpha_2 - 1)$, we find for every $x \in \Omega$

$$|x - x_0| < r + \operatorname{diam}(\Omega) \leq r + (\alpha_2 - 1)r = \alpha_2 r$$

so that $\Omega \subset B(x_0, \alpha_2 r)$. It follows that $\chi_\Omega f = 0$ on $\mathbb{R}^d$ if f satisfies $f = 0$ on $B(x_0, \alpha_2 r)$. By linearity of $T_{\mathbb{R}^d}$ we conclude $T_{\mathbb{R}^d} f = 0$ for all such functions f, so that the weak reverse Hölder estimates are valid for these radii.

If $\operatorname{diam}(\Omega)/(\alpha_2 - 1) \leq R_0$ we can conclude the proof. If not, we appeal to Lemma 3.2.3 and conclude that in the case where Ω is bounded, $T_{\mathbb{R}^d}$ satisfies the weak reverse Hölder estimates for all balls in $\mathbb{R}^d$. By means of Step 1, we conclude that the restriction of $T_{\mathbb{R}^d}$ onto $\mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^q(\mathbb{R}^d; X)$ extends to a bounded operator from $\mathrm{L}^q(\mathbb{R}^d; X)$ into $\mathrm{L}^q(\mathbb{R}^d; Y)$ and that by construction of $T_{\mathbb{R}^d}$ the same is true for $T : \mathrm{L}^q(\Omega; X) \to \mathrm{L}^q(\Omega; Y)$. $\square$

CHAPTER 4

Boundary value problems

Boundary value problems play an eminent role in the study of differential operators on Lipschitz domains in the L^p-setting. For example, if we consider the Laplacian with Dirichlet boundary conditions on a bounded Lipschitz domain Ω, then, by testing the equation

$$-\Delta u = f$$

with the solution u, one directly finds that $f \mapsto \nabla u$ yields a bounded operator on L^2, i.e., the operator $T := \nabla(-\Delta)^{-1}$ is bounded on L^2. One way to derive L^p-estimates on the gradient of u is to appeal to Shen's L^p-extrapolation theorem 3.1.1. To verify the hypothesis of this theorem, for some $p > 2$, one establishes the validity of the weak reverse Hölder estimate

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \leq \mathcal{C}\left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \, \mathrm{d}x\right)^{\frac{1}{2}},$$

whenever u is a weak solution to $-\Delta u = f$ with $f = 0$ on $B(x_0, 3r) \cap \Omega$. In this case, u is harmonic inside $B(x_0, 3r) \cap \Omega$, has vanishing trace on $B(x_0, 3r) \cap \partial\Omega$ and has a non-vanishing trace on $\Omega \cap \partial B(x_0, sr)$ for each $s \in [1, 2]$. Thus, u solves the Dirichlet problem in $\Omega \cap B(x_0, sr)$ with proper boundary data. In order to derive weak reverse Hölder estimates, we will see in Chapter 5, that it is crucial to have a good understanding of solutions to the Dirichlet problem, or, more generally, to certain boundary value problems.

For this purpose, we use Section 4.1 to introduce to boundary value problems for either harmonic functions or functions that solve the homogeneous stationary Stokes (resolvent) equations. Furthermore, we will give an overview over the existing results that are of importance in the treatment of Chapter 5. Our main result of this chapter is Theorem 4.1.9 and is a generalization of results derived by Shen in [86] and by Choe and Kozono in [17].

We use Section 4.2 to introduce briefly to Hankel and Bessel funcions, since for the proof of Theorem 4.1.9, we need one additional estimate that involves these functions and that was not proven by Shen in [89, Sec. 2]. The proof of our theorem will be performed in Section 4.3.

For the whole chapter, the number r_0 from the definition of a bounded Lipschitz domain, Definition 1.3.1, and the Lipschitz character, Definition 1.3.3, will be of importance.

4.1 Boundary value problems with L^2 boundary data

The first boundary value problem, treated here, is the L^2-Neumann problem of the Laplacian, i.e., on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with connected boundary and for every function $g \in \mathrm{L}^2(\partial\Omega)$ with vanishing mean, we seek solutions to the problem

$$(\mathrm{Neu_L}) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = g & \text{non-tangentially on } \partial\Omega \\ (\nabla u)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

Recall that ν denotes the exterior unit normal to $\partial\Omega$. At first, we must clarify the sense in which u is supposed to be a solution to the Neumann problem above.

Definition 4.1.1. Given $g \in \mathrm{L}^2(\partial\Omega)$ with vanishing mean (this will be indicated by $g \in \mathrm{L}^2_0(\partial\Omega)$), we say that u is a solution to $(\mathrm{Neu_L})$ if u is a smooth harmonic function such that ∇u converges non-tangentially σ-a.e. to a function $\boldsymbol{\nabla u}$ on $\partial\Omega$, such that $\langle \nu, \boldsymbol{\nabla u} \rangle = g$ σ-a.e. on $\partial\Omega$, and such that $(\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$.

The other boundary value problems we are considering concern the Stokes and the Stokes resolvent problems. Here, we are more interested in the L^2-Dirichlet problem and in the regularity theory for the L^2-Dirichlet problem. For $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ and

$$g \in \mathrm{L}^2_\nu(\partial\Omega) := \Big\{ g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) : \int_{\partial\Omega} \langle \nu, g \rangle \, \mathrm{d}\sigma = 0 \Big\} \tag{4.1}$$

the L^2-Dirichlet problem for the Stokes (resolvent) problem reads as

$$(\mathrm{Dir}_{\mathrm{S},\lambda}) \qquad \begin{cases} \lambda u - \Delta u + \nabla \phi = 0 & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = g & \text{non-tangentially on } \partial\Omega \\ (u)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

Note that we only say *Stokes problem* if we mean $\lambda = 0$ and that we say *Stokes resolvent problem* if $\lambda \neq 0$.

Definition 4.1.2. Given $g \in \mathrm{L}^2_\nu(\partial\Omega)$, we say that u and ϕ solve $(\mathrm{Dir}_{\mathrm{S},\lambda})$ if u and ϕ are smooth and satisfy all conditions of $(\mathrm{Dir}_{\mathrm{S},\lambda})$.

Having formulated the L^2-Dirichlet problem, one could ask for a regularity theory, i.e., the investigation of the question whether u and ϕ possess better properties if the data lie in a better space; in our case if g lies additionally in $\mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$. This will be done by resolving the problem

$$(\mathrm{Reg}_{\mathrm{S},\lambda}) \qquad \begin{cases} \lambda u - \Delta u + \nabla \phi = 0 & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = g & \text{non-tangentially on } \partial\Omega \\ (u)^*, (\nabla u)^*, (\phi)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

Here, u and ϕ are solutions to the L^2-Dirichlet problem, but ∇u and ϕ satisfy additional non-tangential behavior.

One way to solve the problems above is via the methods of single and double layer potentials. We will give a short glimpse into the definition of single layer potentials in the next subsection.

4.1.1 A glimpse into single layer potentials

To introduce single layer potentials, we need to introduce fundamental solutions to the partial differential equations presented above. For the Laplacian, it is classical that a fundamental solution is given by

$$\mathrm{G}(x;0) := \begin{cases} \frac{1}{\omega_d(d-2)} \, |x|^{2-d}, & \text{if } d \geq 3 \\ -\frac{1}{2\pi} \log(|x|), & \text{if } d = 2. \end{cases} \tag{4.2}$$

Here, ω_d denotes the surface measure of the unit ball in $\mathbb{R}^d$.

Following the treatise of Verchota [95], see also [96] of the same author, given $f \in \mathrm{L}^2(\partial\Omega)$ the *single layer potential of the Laplacian* is defined as

$$[\mathcal{S}_{\mathrm{L}} f](x) := \int_{\partial\Omega} \mathrm{G}(x-y;0) f(y) \, \mathrm{d}\sigma(y) \qquad (x \in \mathbb{R}^d \setminus \partial\Omega).$$

Now, we can formulate the resolution of the L^2-Neumann problem of the Laplacian. For the proof of the three and higher dimensional case, the reader may consult [96, Cor. 3.4]. The L^2-estimate of $(\nabla u)^*$ by f follows by combining [96, Lem. 1.3 & Thm. 3.3]. The two dimensional case follows by [96, Thm. 4.9].

Theorem 4.1.3 (Verchota). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with connected boundary and let $\{\Gamma(q)\}_{q \in \partial\Omega}$ be a regular family of cones. Given $g \in \mathrm{L}^2_0(\partial\Omega)$ there exists a solution u to* $(\mathrm{Neu}_{\mathrm{L}})$*, which is unique up to an additional constant and which can be written as $u = \mathcal{S}_{\mathrm{L}} f$ for some $f \in \mathrm{L}^2_0(\partial\Omega)$. Moreover, there exists a constant $C > 0$ depending only on d, the Lipschitz character of Ω, and the regular family of cones, such that*

$$\begin{aligned} \|(\nabla u)^*\|_{\mathrm{L}^2(\partial\Omega)} &\leq C \|f\|_{\mathrm{L}^2(\partial\Omega)} \\ \|f\|_{\mathrm{L}^2(\partial\Omega)} &\leq C \|g\|_{\mathrm{L}^2(\partial\Omega)}, \end{aligned}$$

where the non-tangential maximal function $()^*$ is defined via $\{\Gamma(q)\}_{q \in \partial\Omega}$.

Remark 4.1.4. (1) On Lipschitz domains, the L^2-Neumann problem for the Laplacian was first solved by Jerison and Kenig [54] by different methods. Note that in comparison to Verchota, Jerison and Kenig do not derive a representation formula.

(2) Taking the regular family of cones described in Remark 1.3.16 (3), we conclude with this special choice, that the constant C in Theorem 4.1.3 depends only on d, r_0, and the Lipschitz character of Ω.

Following the classical treatise of Ladyzhenskaya [62] or the work of Fabes, Kenig, and Verchota [28], for $d \geq 3$, the matrix of fundamental solutions of the Stokes problem ($\lambda = 0$) is given by

$$\Gamma_{jk}(x;0) := \frac{1}{2\omega_d}\Big(\frac{1}{d-2}\frac{\delta_{jk}}{|x|^{d-2}} + \frac{x_j x_k}{|x|^d}\Big) \qquad (x \in \mathbb{R}^d \setminus \{0\}) \tag{4.3}$$

and the *corresponding pressure vector* by

$$\Phi(x) := \frac{1}{\omega_d}\frac{1}{|x|^d}x = -\nabla \mathrm{G}(x;0) \qquad (x \in \mathbb{R}^d \setminus \{0\}). \tag{4.4}$$

In (4.3), δ_{jk} denotes Kronecker's delta. If $\Gamma(x;0)$ denotes the matrix $(\Gamma_{jk}(x;0))_{j,k=1}^d$, then, for $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, the *single layer potential for the Stokes problem* is given by

$$[\mathcal{S}_0 f](x) := \int_{\partial\Omega} \Gamma(x-y;0) f(y)\, \mathrm{d}\sigma(y) \qquad (x \in \mathbb{R}^d \setminus \partial\Omega) \tag{4.5}$$

and the corresponding pressure is given by

$$\phi(x) := \int_{\partial\Omega} \langle \Phi(x-y), f(y)\rangle\, \mathrm{d}\sigma(y) \qquad (x \in \mathbb{R}^d \setminus \partial\Omega). \tag{4.6}$$

On bounded Lipschitz domains with connected boundary, the L^2-Dirichlet problem for the Stokes problem was resolved by Fabes, Kenig, and Verchota [28, Thm. 3.9] by use of the method of double layer potentials. In the case of smooth boundaries, this was resolved by Ladyzhenskaya in [62, Ch. 3] by means of the same method.

Theorem 4.1.5 (Fabes, Kenig, Verchota). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary. Given $g \in \mathrm{L}^2_\nu(\partial\Omega)$ the Dirichlet problem* $(\mathrm{Dir}_{\mathrm{S},0})$ *is uniquely solvable (the pressure is unique up to constants).*

More interesting to us will be the regularity theory for the L^2-Dirichlet problem, which can be found in [28, Thm. 4.15].

Theorem 4.1.6 (Fabes, Kenig, Verchota). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary and let $\{\Gamma(q)\}_{q\in\partial\Omega}$ be a regular family of cones. Given $g \in \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d) \cap \mathrm{L}^2_\nu(\partial\Omega)$ the problem* $(\mathrm{Reg}_{\mathrm{S},0})$ *is uniquely solvable (the pressure is unique up to constants). This solution is given by $u = \mathcal{S}_0 f$ for some $f \in \mathrm{L}^2_\nu(\partial\Omega)$ and the pressure ϕ is given by* (4.6)*. Moreover, there exists a constant $C > 0$ depending only on d, r_0, the Lipschitz character of Ω, and the regular family of cones, such that*

$$\begin{aligned} \|(u)^*\|_{\mathrm{L}^2(\partial\Omega)} + \|(\nabla u)^*\|_{\mathrm{L}^2(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^2(\partial\Omega)} &\leq C\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \\ \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} &\leq C\|g\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \end{aligned}$$

To understand the problems $(\mathrm{Dir}_{\mathrm{S},\lambda})$ and $(\mathrm{Reg}_{\mathrm{S},\lambda})$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, we introduce the matrix of fundamental solutions of the Stokes resolvent problem in the following.

Fix an angle $\theta \in [0, \pi)$ with $\lambda \in \mathrm{S}_\theta$ and write $\lambda = r\mathrm{e}^{\mathrm{i}\tau}$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$. Now, define the number

$$k := \sqrt{r}\mathrm{e}^{\mathrm{i}(\pi+\tau)/2} \tag{4.7}$$

so that $k^2 = -\lambda$. Recall that S_θ is defined as the sector in the complex plane symmetric about the positive real axis of angle 2θ.

In three and more dimensions, we proceed by showing how to construct the fundamental solution for the Stokes resolvent problem from the fundamental solution of the scalar Helmholtz equation $\lambda u - \Delta u = 0$ in $\mathbb{R}^d$. Following McLean [70, p. 282], this fundamental solution is given by

$$\mathrm{G}(x; \lambda) := \frac{\mathrm{i}}{4(2\pi)^{\frac{d-2}{2}}} \frac{1}{|x|^{d-2}} (k\,|x|)^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(k\,|x|), \tag{4.8}$$

where $\mathrm{H}^{(1)}_\nu$ is the Hankel function of the first kind, which has the representation

$$\mathrm{H}^{(1)}_\nu(z) = \frac{2^{\nu+1}\mathrm{e}^{\mathrm{i}(z-\nu\pi)}z^\nu}{\mathrm{i}\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{2zis} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s \tag{4.9}$$

if $\nu > -\frac{1}{2}$ and $0 < \arg(z) < \pi$, see Lebedev [63, p. 120]. Here, Γ is the usual Γ-function that should not be confused with the fundamental solution of the Stokes (resolvent) problem, where one always finds the resolvent parameter in the argument. Note that for $\nu = 1/2$

$$\mathrm{H}^{(1)}_{\frac{1}{2}}(z) = \frac{2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}(z-\frac{\pi}{2})}z^{\frac{1}{2}}}{\mathrm{i}\sqrt{\pi}} \int_0^\infty \mathrm{e}^{2zis}\,\mathrm{d}s = \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}(z-\frac{\pi}{2})}z^{\frac{1}{2}}}{\sqrt{\pi}}\frac{1}{z}.$$

Consequently, for $d = 3$, the fundamental solution of the scalar Helmholtz equation is given by

$$\mathrm{G}(x;\lambda) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi\,|x|}. \tag{4.10}$$

Let us turn our attention again towards the fundamental solution to the Stokes resolvent problem. Note that if one applies the divergence to the first equation of the Stokes resolvent problem

$$\text{(SRP)} \qquad \begin{cases} \lambda u - \Delta u + \nabla\phi = 0 & \text{on } \mathbb{R}^d \setminus \{0\} \\ \operatorname{div}(u) = 0 & \text{on } \mathbb{R}^d \setminus \{0\} \end{cases}$$

one gets by Schwarz' theorem that ϕ must be harmonic. Thus, a fundamental solution of the pressure could simply be the fundamental solution of the Laplacian $\mathrm{G}(x;0)$ or any derivative of this function. As in (4.4), we take $\Phi_\beta(x) = -\partial_\beta \mathrm{G}(x;0)$ as the fundamental solution of the pressure. Because β ranges between $1,\ldots,d$, we can choose d different fundamental solutions for the velocity u. For fixed β denote the fundamental solution of the velocity u by $(\Gamma_{\alpha\beta}(x;\lambda))_{\alpha=1}^d$. Because

$$\begin{cases} \lambda\frac{1}{\lambda}\nabla\Phi_\beta - \Delta\frac{1}{\lambda}\nabla\Phi_\beta = \nabla\Phi_\beta \\ \frac{1}{\lambda}\operatorname{div}(\nabla\Phi_\beta) = 0 \end{cases}$$

one can replace the pressure in (SRP) by the left-hand side above, so that $(\Gamma_{\alpha\beta}(\cdot;\lambda))_{\alpha=1}^d$ and Φ_β solve (SRP) if $\Lambda_{\alpha\beta} := \Gamma_{\alpha\beta}(\cdot;\lambda) + \partial_\alpha\Phi_\beta/\lambda$ solves

$$\begin{cases} \lambda\Lambda_{\cdot\beta} - \Delta\Lambda_{\cdot\beta} = 0 \\ \operatorname{div}(\Lambda_{\cdot\beta}) = 0, \end{cases}$$

where $\Lambda_{\cdot\beta}$ is understood to be the vector $(\Lambda_{\alpha\beta})_{\alpha=1}^d$. The first equation above suggests to take a linear combination of fundamental solutions of the scalar Helmholtz equation. However, this linear combination must be chosen such that the terms arising in the divergence cancel. Since

$$0 = \partial_\beta \mathrm{G}(x;\lambda) - \frac{1}{\lambda}\partial_\beta \Delta \mathrm{G}(x;\lambda) = \operatorname{div}\Big(\mathrm{G}(x;\lambda)e_\beta - \frac{1}{\lambda}\nabla\partial_\beta \mathrm{G}(x;\lambda)\Big),$$

where e_β denotes the βth unit vector of $\mathbb{R}^d$, this is achieved by taking

$$\Lambda_{\alpha\beta}(x) := \mathrm{G}(x;\lambda)\delta_{\alpha\beta} - \frac{1}{\lambda}\partial_\alpha\partial_\beta \mathrm{G}(x;\lambda).$$

This yields a choice for the *fundamental matrix of the Stokes resolvent problem*. Namely,

$$\Gamma_{\alpha\beta}(x;\lambda) := \mathrm{G}(x;\lambda) - \frac{1}{\lambda}\partial_\alpha\partial_\beta\{\mathrm{G}(x;\lambda) - \mathrm{G}(x;0)\}. \tag{4.11}$$

With $\Gamma(\cdot;\lambda)$ denoting the matrix $(\Gamma_{\alpha\beta}(\cdot;\lambda))_{\alpha,\beta=1}^d$ the *single layer potential for the Stokes resolvent problem* is defined for $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ by

$$[\mathcal{S}_\lambda f](x) := \int_{\partial\Omega} \Gamma(x-y;\lambda) f(y)\,\mathrm{d}\sigma(y) \qquad (x \in \mathbb{R}^d \setminus \partial\Omega).$$

Note that the definition of $\Gamma(\cdot;\lambda)$ shows that it is radially symmetric and that it has the scaling behavior

$$\Gamma(rx;\lambda) = r^{2-d}\Gamma(x;r^2\lambda) \qquad (x \in \mathbb{R}^d \setminus \{0\}, r > 0). \tag{4.12}$$

In the treatment of the L^2-Dirichlet problem of the Stokes resolvent problem, Shen used another type of non-tangential maximal function in [89], namely, for $p \in \partial\Omega$ and a function $f : \Omega \to \mathbb{C}^d$, he defined

$$[N_a f](p) := \sup\{|f(x)| : |x-p| < (1+a)\operatorname{dist}(x,\partial\Omega)\}, \tag{4.13}$$

where $a > 0$ is a fixed number. A comparison of the non-tangential maximal functions $()^*$ and N_a is postponed until the end of this section. With the non-tangential maximal function above, he resolved the L^2-Dirichlet problem for the Stokes resolvent, see [89, Thm. 5.5].

Theorem 4.1.7. (Shen) *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary, $\theta \in [0, \pi)$, and $\lambda \in \mathrm{S}_\theta$. Given $g \in \mathrm{L}^2_\nu(\partial\Omega)$ the L^2-Dirichlet problem for the Stokes resolvent* $(\mathrm{Dir}_{\mathrm{S},\lambda})$*, where the non-tangential maximal function is replaced by N_a, has a unique solution u and the pressure ϕ is unique up to constants. Moreover, there exists a constant $C > 0$ depending only on d, θ, and the Lipschitz character of Ω, such that*

$$\|N_a u\|_{\mathrm{L}^2(\partial\Omega)} \leq C\|g\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}.$$

Remark 4.1.8. The remarkable achievement in the theorem above is that the constant C does not depend on the resolvent parameter λ.

The theorem below establishes the regularity theory to the L^2-Dirichlet problem of the Stokes resolvent problem $(\mathrm{Reg}_{\mathrm{S},\lambda})$. For λ on the imaginary axis, this was already resolved by Shen in the proof of [86, Thm. 5.2.1] and by Choe and Kozono in [17, Thm. 3.7]. However, for $\lambda \in \mathrm{S}_\theta$, $\theta \in [0, \pi)$, one needs the results of [89] and one further estimate on the difference of the matrices of fundamental solutions to the Stokes resolvent problem and the Stokes problem. This estimate as well as the proof of the theorem are presented in Section 4.3. With the proper replacements, the proof follows closely the proofs of the authors above.

Theorem 4.1.9. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary, $\theta \in [0, \pi)$, $\lambda \in \mathrm{S}_\theta$, and let $\{\Gamma(q)\}_{q \in \partial\Omega}$ be a regular family of cones. Then, for all $g \in \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d) \cap \mathrm{L}^2_\nu(\Omega)$ the problem* $(\mathrm{Reg}_{\mathrm{S},\lambda})$ *has a unique solution u and the pressure ϕ is unique up to constants. The velocity u can be represented as the single layer potential $\mathcal{S}_\lambda f$ for some function $f \in \mathrm{L}^2_\nu(\partial\Omega)$ and the pressure is given by* (4.6)*. Moreover, there exists a constant $C_1 > 0$ depending only on d, θ, r_0, the Lipschitz character of Ω, and the regular family of cones, such that*

$$|\lambda|^{1/2} \|(u)^*\|_{\mathrm{L}^2(\partial\Omega)} + \|(\nabla u)^*\|_{\mathrm{L}^2(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^2(\partial\Omega)} \leq C_1\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)},$$

a constant $C_2 > 0$ depending on the same quantities as C_1 and additionally on $\operatorname{diam}(\Omega)$*, such that for all λ with $|\lambda| > \mathrm{e}^{-2}\operatorname{diam}(\Omega)^{-2}$*

$$\begin{aligned}&\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}\\ &\qquad \leq C_2\Big\{\|\nabla_{\tan} g\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} + |\lambda|^{1/2}\|g\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} + |\lambda|\,\|\langle \nu, g\rangle\|_{(\mathrm{W}^{1,2}(\partial\Omega))^*}\Big\},\end{aligned}$$

and a constant $C_3 > 0$ *depending on the same quantities as* C_1*, such that for all* λ *with* $|\lambda| \leq \mathrm{e}^{-2} \operatorname{diam}(\Omega)^{-2}$

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C_3 \, |\lambda|^{1/2} \, \|g\|_{\mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)}.$$

Remark 4.1.10. In the theorem above, $(\mathrm{W}^{1,2}(\partial\Omega))^*$ denotes the anti-dual space of $\mathrm{W}^{1,2}(\partial\Omega)$ and $\mathrm{L}^2(\partial\Omega)$ embeds into $(\mathrm{W}^{1,2}(\partial\Omega))^*$ by means of the usual identification

$$f \quad \leftrightarrow \quad \Big[\mathrm{W}^{1,2}(\partial\Omega) \ni u \mapsto \int_{\partial\Omega} f \overline{u} \, \mathrm{d}\sigma\Big].$$

Comparing the resolution to the L^2-Dirichlet problem for the Stokes resolvent with the resolutions to the other boundary value problems, one notes that two different non-tangential maximal functions are used. In the remainder of this section, we show that one can interchange the different non-tangential maximal functions in those results, where the solution has a representation as a single layer potential.

For this purpose, define for $a > 0$

$$\gamma_a(p) := \{x \in \Omega : |x - p| < (1 + a) \operatorname{dist}(x, \partial\Omega)\} \qquad (p \in \partial\Omega), \tag{4.14}$$

which is a “cone” at p with a certain opening angle. Note that there are essential differences between cones $\Gamma(p)$ of a regular family of cones and the cones $\gamma_a(p)$. For example, if a is too small (depending on the Lipschitz constant of the domain), then $\gamma_a(p)$ could be empty for certain $p \in \partial\Omega$. An example for this is a domain Ω, which itself is a cone with vertex at zero and which is symmetric about the x_d-axis. If the opening angle of this cone is $\alpha \in (0, \pi)$ and if $x \in \Omega$, then

$$|x| = \frac{\operatorname{dist}(x, \partial\Omega)}{\sin(\beta)}$$

for some $\beta \in (0, \alpha/2]$. If $0 < a \leq 1/\sin(\alpha/2) - 1$, then we find for every $x \in \Omega$

$$|x| \geq (1 + a) \operatorname{dist}(x, \partial\Omega),$$

so that $\gamma_a(0) = \emptyset$.

Another difference between $\gamma_a(p)$ and $\Gamma(p)$ is that every point $x \in \Omega$ is contained in at least one of the cones $\gamma_a(p)$ (x is always contained in $\gamma_a(p)$ with $p \in \partial\Omega$ satisfying $\operatorname{dist}(x, \partial\Omega) = |x - p|$). This does not have to be valid for the cones $\Gamma(p)$ as they only exist locally in a Lipschitz cylinder.

These two examples show that one cannot hope to compare the non-tangential maximal functions $(f)^*$ and $N_a f$ in a pointwise manner, compare Definition 1.3.17 and (4.13) to recall the definitions.

Even though the cones $\gamma_a(p)$ could be empty for some $p \in \partial\Omega$ and non-empty if a is large enough, one can still compare $N_a f$ and $N_b f$ for all $a, b > 0$ in $\mathrm{L}^2(\partial\Omega)$, as the following proposition shows.

Proposition 4.1.11. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then, for all $b > a > 0$ there exists a constant $C > 0$ depending only on a, b, and the Lipschitz character of Ω such that for all measurable $f : \Omega \to \mathbb{C}^N$ with $N_a f \in \mathrm{L}^2(\partial\Omega)$*

$$\|N_a f\|_{\mathrm{L}^2(\partial\Omega)} \leq \|N_b f\|_{\mathrm{L}^2(\partial\Omega)} \leq C \|N_a f\|_{\mathrm{L}^2(\partial\Omega)}.$$

Proof. By definition it is clear that $[N_a f](p) \leq [N_b f](p)$ holds for all $p \in \partial\Omega$, so that only the second inequality has to be considered. This inequality was proven by Barton [7, Lem. 3.2] in two dimensions, and the same proof literally applies to all space dimensions. See Calderón and Torchinsky [15, Lem. 2.2] for the same result on the half-space in any dimension. $\square$

For the remainder of this section, recall the numbers M and r_0 from Definition 1.3.1. Moreover, recall the notation concerning regular families of cones, which was fixed in Definition 1.3.15 and Remark 1.3.16 (3).

Lemma 4.1.12. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $\{\Gamma(q)\}_{q \in \partial\Omega}$ be a regular family of cones. Then there exists $a > 0$ such that*

$$\Gamma(q) \subset \gamma_a(q) \qquad (q \in \partial\Omega).$$

Proof. Let $q \in \partial\Omega$. By Definition 1.3.15, there exists $x_i \in \partial\Omega$, $\widetilde{r} > 0$, and a rotation $\widetilde{R}_{x_i}^{-1}$ such that

$$q \in \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5).$$

Moreover, by the same definition, with the number ν defined by (1.10), we have that

$$\widetilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(\nu\widetilde{r}) = D_{\widetilde{\eta}_{x_i}}(\nu\widetilde{r}).$$

By definition of this number ν, one verifies with a simple calculation that for all $y \in \{x_i\} + \widetilde{R}_{x_i}^{-1} D(\widetilde{r})$

$$\operatorname{dist}(y, \partial\Omega) = \operatorname{dist}(y, \partial\Omega \cap [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\nu\widetilde{r})]).$$

By Definition 1.3.15 we have $\Gamma(q) \subset \{x_i\} + \widetilde{R}_{x_i}^{-1} D(\widetilde{r})$, so that the identity for the distances holds in particular for all points in $\Gamma(q)$. Moreover, by (1.11) there exists a cone β_i such that $\Gamma(q) \subset \{q\} + R_{x_i}^{-1}\beta_i$. Recalling also the cone γ_i from Definition 1.3.15, we derive by elementary trigonometry (calculate the distance of y to the lateral surface of $\{q\} + R_{x_i}^{-1}\gamma_i$) that

$$\operatorname{dist}(y, \partial\Omega \cap [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\nu\widetilde{r})]) > |y - q| \sin((\vartheta_{\gamma_i} - \vartheta_{\beta_i})/2).$$

Note that (1.11) ensures that $\vartheta_{\gamma_i} > \vartheta_{\beta_i}$. Consequently, $y \in \gamma_{a_i}(q)$ with $1/(1 + a_i) := \sin((\vartheta_{\gamma_i} - \vartheta_{\beta_i})/2)$. Since in the definition of regular families of cones only finitely many cones β_i and γ_i occur, we conclude the proof by taking a to be maximum of the a_i. □

Combining the previous lemma with the previous proposition, we arrive at the following corollary.

Corollary 4.1.13. *The conclusion of Theorem 4.1.7 remains valid with the usual non-tangential maximal function defined in Definition 1.3.17.*

For further use, we record the following lemma.

Lemma 4.1.14. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Then, there exists a constant $C > 0$, depending only on d and M, such that*

$$\sigma(B(q, r) \cap \partial\Omega) \leq Cr^{d-1} \qquad (0 < r < r_0, q \in \partial\Omega).$$

Furthermore, there exists a constant $C > 0$, depending only on d and the Lipschitz character of Ω, such that

$$\sigma(\partial\Omega) \leq Cr_0^{d-1}.$$

Proof. Notice that by the definition of the surface measure σ, see (1.6), for all $q \in \partial\Omega$ and $0 < r \leq r_0$ the inequality

$$\sigma(B(q,r) \cap \partial\Omega) \leq \frac{\omega_{d-1}[1+M^2]^{1/2}}{d} r^{d-1}$$

holds, where ω_{d-1} is the surface measure of the $(d-1)$-dimensional unit ball.

To prove the second inequality, note that if we cover $\partial\Omega$ by the sets $(U_{x_1,r_0})_{i=1}^n$, then (1.6) shows that

$$\sigma(\partial\Omega) \leq \frac{\omega_{d-1}[1+M^2]^{1/2} n}{d} r_0^{d-1}. \qquad \square$$

Given a regular family of cones $\{\Gamma(q)\}_{q\in\partial\Omega}$, recall the points $x_1, \ldots, x_{n_0}$, $\widetilde{r} > 0$, the rotations $\widetilde{R}_{x_1}, \ldots, \widetilde{R}_{x_{n_0}}$, the Lipschitz continuous functions $\widetilde{\eta}_{x_1}, \ldots \widetilde{\eta}_{x_{n_0}}$, and the cones $\alpha_1, \ldots, \alpha_{n_0}$. Recall the height h_i of α_i measured from the base to the vertex along the axis of revolution the opening angle ϑ_{α_i}. Finally, define

$$\mathfrak{d} := \operatorname{dist}\Big(\partial\Omega, \Omega \setminus \Big[\bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5)\Big]\Big).$$

The following lemma shows, that under certain circumstances, one can control $N_a u$ by $(u)^*$ and an additional term.

Lemma 4.1.15. *Let $a > 0$ and $\{\Gamma(q)\}_{q\in\partial\Omega}$ be a regular family of cones. Assume that given a function $u : \Omega \to \mathbb{C}^N$ with $(u)^* \in \mathrm{L}^2(\partial\Omega)$, there exists $f \in \mathrm{L}^2(\partial\Omega)$, $\mathcal{C} > 0$, and $l \in \mathbb{N}_0$ such that*

$$|u(x)| \leq \mathcal{C} \int_{\partial\Omega} |x-y|^{-l} |f(y)| \, \mathrm{d}\sigma(y).$$

Then, there exists a constant $C > 0$, depending only on l, $\mathcal{C}$, the minimum of the heights h_i, the minimum of the opening angles ϑ_{α_i}, the maximum of the Lipschitz constants $\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$, $\mathfrak{d}$, $\widetilde{r}$, n_0, and a such that

$$\|N_a u\|_{\mathrm{L}^2(\partial\Omega)} \leq C\Big\{\|(u)^*\|_{\mathrm{L}^2(\partial\Omega)} + \|f\|_{\mathrm{L}^2(\partial\Omega)}\Big\}.$$

Proof. Let $y \in \gamma_a(q)$ for some $q \in \partial\Omega$ with $\operatorname{dist}(y, \partial\Omega) \leq \varepsilon$, where $\varepsilon > 0$ is a number to be determined during the proof. Let $x_1, \dots, x_{n_0} \in \partial\Omega$, $\widetilde{r} > 0$, $\widetilde{R}_{x_1}, \dots, \widetilde{R}_{x_{n_0}}$, and $\widetilde{\eta}_{x_1}, \dots, \widetilde{\eta}_{x_{n_0}}$ be the quantities from Definition 1.3.15. Then, by the same definition

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5).$$

Define

$$\mathfrak{d} := \operatorname{dist}\Big(\partial\Omega, \Omega \setminus \Big[\bigcup_{i=1}^{n_0} \{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5)\Big]\Big)$$

and the coordinate function analogously to (1.5) as

$$\widetilde{\Phi}_{x_i,\widetilde{r}}(z') := x_i + \widetilde{R}_{x_i}^{-1} \begin{pmatrix} z' \\ \widetilde{\eta}_{x_i}(z') \end{pmatrix}.$$

It follows that $\mathfrak{d} > 0$. If $\varepsilon \leq (1 + a)^{-1}\delta$, then by definition of $\gamma_a(q)$, cf. (4.14), we have $|y - q| < \mathfrak{d}$. It follows that there exists $1 \leq i \leq n_0$ such that

$$y \in \Omega \cap [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(4\widetilde{r}/5)].$$

Next, note that due to the definition of ν in (1.10), we find by an elementary calculation that

$$\operatorname{dist}(y, \partial\Omega) = \operatorname{dist}(y, \partial\Omega \cap [\{x_i\} + \widetilde{R}_{x_i}^{-1} D(\nu\widetilde{r})]).$$

Now, we simplify the notation a little bit and assume without loss of generality that

$$y \in D_{\widetilde{\eta}_{x_i}}(4\widetilde{r}/5) \quad \text{and} \quad \operatorname{dist}(y, \partial\Omega) = \operatorname{dist}(y, I_{\widetilde{\eta}_{x_i}}(\nu\widetilde{r})).$$

Next, we deduce by some trigonometry that

$$\cos(\arctan(\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}))\,|y_d - \widetilde{\eta}_{x_i}(y')| \leq \operatorname{dist}(y, I_{\widetilde{\eta}_{x_i}}(\nu\widetilde{r})).$$

Combining this with the previous equality, we deduce

$$\cos(\arctan(\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}))\,|y_d - \widetilde{\eta}_{x_i}(y')| \leq (1+a)\varepsilon. \tag{4.15}$$

Let $h_i > 0$ denote the height of α_i and define $h := \min_{i=1}^{n_0} h_i$. It follows that y is contained in the regular cone $\Gamma((y', \widetilde{\eta}_{x_i}(y')))$ if the following condition is imposed on ε

$$\varepsilon \leq \frac{h\cos(\arctan(\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}))}{2(1+a)}. \tag{4.16}$$

It is now the aim to show that y is contained in all regular cones in a certain neighborhood of $(y', \widetilde{\eta}_{x_i}(y'))$. Recall that ϑ_{α_i} denotes the opening angle of α_i. Define $\vartheta_\alpha := \min_{i=1}^n \vartheta_{\alpha_i}$ and

$$\delta := \min\left\{\frac{4\widetilde{r}}{5}, \frac{|y_d - \widetilde{\eta}_{x_i}(y')|\tan(\vartheta_\alpha/2)}{1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\tan(\vartheta_\alpha/2)}\right\}$$

and let $z' \in B'(y', \delta) \cap B'(0, 4\widetilde{r}/5)$.

First of all, note that, by (1.11), the regular cone at $(z', \widetilde{\eta}_{x_i}(z'))$ contains $\{(z', \widetilde{\eta}_{x_i}(z'))\} + \alpha_i$. Second, we find that $y_d > \widetilde{\eta}_{x_i}(z')$ since $y_d > \widetilde{\eta}_{x_i}(y')$ and

$$\begin{aligned}|\widetilde{\eta}_{x_i}(y') - \widetilde{\eta}_{x_i}(z')| &\leq \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\delta \\ &\leq \frac{\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\tan(\vartheta_\alpha/2)}{1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\tan(\vartheta_\alpha/2)}\,|y_d - \widetilde{\eta}_{x_i}(y')|\,.\end{aligned}$$

Moreover, combining this inequality with (4.15) and (4.16) yields

$$|y_d - \widetilde{\eta}_{x_i}(z')| \leq \left[1 + \frac{\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\tan(\vartheta_\alpha/2)}{1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\tan(\vartheta_\alpha/2)}\right]|y_d - \widetilde{\eta}_{x_i}(y')| < h,$$

so that $y_d < \eta_{x_i}(z') + h$. Third, by elementary trigonometry, we find that under the previous two conditions y lies inside $\Gamma((z', \widetilde{\eta}_{x_i}(z')))$ if

$$|y' - z'| < \tan\left(\frac{\vartheta_\alpha}{2}\right)|y_d - \widetilde{\eta}_{x_i}(z')|\,.$$

This is satisfied, since by the choice of δ

$$\begin{aligned}|y_d - \widetilde{\eta}_{x_i}(z')| &\geq |y_d - \widetilde{\eta}_{x_i}(y')| - \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\, |y' - z'| \\ &> \frac{1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \tan(\vartheta_\alpha/2)}{\tan(\vartheta_\alpha/2)}\, |y' - z'| \\ &\qquad - \|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\, |y' - z'| \\ &= \frac{|y' - z'|}{\tan(\vartheta_\alpha/2)}.\end{aligned}$$

We conclude that $y \in \Gamma((z', \widetilde{\eta}_{x_i}(z')))$ for all $z' \in B'(y', \delta) \cap B'(0, 4\widetilde{r}/5)$. By the choice of δ, we directly see that $\delta \leq 4\widetilde{r}/5$, so that $B'(y', \delta) \cap B'(0, 4\widetilde{r}/5)$ contains a ball B' of radius $\delta/4$. What we have proven so far, is that

$$|u(y)| \leq (u)^*((z', \widetilde{\eta}_{x_i}(z'))) \qquad (z' \in B').$$

Integrating this inequality yields

$$|u(y)| \leq \fint_{B'} (u)^*((z', \widetilde{\eta}_{x_i}(z')))\, \mathrm{d}z'.$$

Next, choosing $\varepsilon \leq \widetilde{r}/[5(1+a)]$, we conclude that $q \in I_{\widetilde{\eta}_{x_i}}(\widetilde{r})$ since $y \in \gamma_a(q)$. Thus, using that $y \in \gamma_a(q)$ again, yields that

$$|y' - q'| \leq (1 + a)\, |y_d - \widetilde{\eta}_{x_i}(y')| =: \delta'.$$

We deduce

$$|u(y)| \leq 4^{d-1}(1 + \delta'/\delta)^{d-1} \fint_{B'(q', \delta + \delta')} (u)^*((z', \widetilde{\eta}_{x_i}(z')))\, \mathrm{d}z'.$$

The choice of ε reveals that $B'(q', \delta' + \delta) \subset \mathrm{B}'(0, 2\widetilde{r})$, so that

$$\leq 4^{d-1}(1 + \delta'/\delta)^{d-1}[\mathrm{M}_{B'(0,2r_0)}(u)^*((\cdot, \widetilde{\eta}_{x_i}(\cdot)))](q').$$

Here, $\mathrm{M}_{B'(0,2\widetilde{r})}$ denotes the localized maximal operator on $B'(0, 2\widetilde{r}) \subset \mathbb{R}^{d-1}$, cf. (3.2). Moreover, the definitions of δ and δ' together with show that the number $[1 + \delta'/\delta]^{d-1}$ is bounded by a constant $C > 0$ depending only on d, a, $\widetilde{r}$, $\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$, and ϑ_α. Calculating the $\mathrm{L}^2(\partial\Omega)$-norm of $N_a u$ then shows

$$\begin{aligned}
\|N_a u\|_{\mathrm{L}^2(\partial\Omega)} \leq \max_{i=1,\dots,n_0} &\Big[1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|^2_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\Big]^{\frac{1}{2}} \\
&\cdot \bigg(\sum_{i=1}^{n_0} \int_{B'(0,4\widetilde{r}/5)} |[N_a u](\widetilde{\Phi}_{x_i,\widetilde{r}}(z'))|^2 \,\mathrm{d}z'\bigg)^{\frac{1}{2}} \\
\leq C \max_{i=1,\dots,n_0} &\Big[1 + \|\nabla_{y'}\widetilde{\eta}_{x_i}\|^2_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}\Big]^{\frac{1}{2}} \\
&\cdot \bigg\{\bigg(\sum_{i=1}^{n_0} \int_{B'(0,2\widetilde{r})} |[\mathrm{M}_{B'(0,2\widetilde{r})}(u)^*(\widetilde{\Phi}_{x_i,\widetilde{r}}(\cdot))](y')|^2 \,\mathrm{d}y'\bigg)^{\frac{1}{2}} \\
&+ n_0^{\frac{1}{2}} \Big[\frac{\omega_{d-1}\widetilde{r}^{d-1}}{d-1}\Big]^{\frac{1}{2}} \sup_A |u| \bigg\},
\end{aligned}$$

where A is given by

$$A := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

Now, the first term can be handled by appealing to the boundedness of the localized maximal operator on $\mathrm{L}^2(B'(0,2\widetilde{r}))$, see Proposition 3.2.1, and by using that

$$\bigg(\sum_{i=1}^{n_0} \int_{B'(0,2\widetilde{r})} |(u)^*(\widetilde{\Phi}_{x_i,2\widetilde{r}}(y'))|^2 \,\mathrm{d}y'\bigg)^{\frac{1}{2}} \quad \text{and} \quad \|(u)^*\|_{\mathrm{L}^2(\partial\Omega)}$$

are equivalent with implicit constants depending only on $\|\nabla_{y'}\widetilde{\eta}_{x_i}\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})}$ and n_0, see Nečas [79, Lem. 3.1.2]. The second term is handled by the assumption of the lemma

$$\begin{aligned}
|u(x)| &\leq \mathcal{C} \int_{\partial\Omega} |x-y|^{-l} f(y) \,\mathrm{d}\sigma(y) \\
&\leq \mathcal{C}\varepsilon^{-l} \int_{\partial\Omega} |f(y)| \,\mathrm{d}\sigma(y).
\end{aligned}$$

An application of Hölder's inequality together with Lemma 4.1.14 concludes the proof. □

4.2 A brief digression on Hankel and Bessel functions

In order to show regularity for the L^2-Dirichlet problem of the Stokes resolvent, we will need estimates for the difference of its fundamental matrix $\Gamma(\cdot;\lambda)$ and the fundamental matrix of the Stokes problem $\Gamma(\cdot;0)$. For a thorough investigation of $\Gamma(\cdot;\lambda)$ as well as estimates on the difference $\Gamma(\cdot;\lambda)-\Gamma(\cdot;0)$, see Shen [89, Sec. 2]. However, we will need some additional estimates on $\Gamma(\cdot;\lambda)-\Gamma(\cdot;0)$, which were not proven in [89]. Furthermore, as Shen spares the details of the derivation of the asymptotic expansion of certain Hankel functions of the first kind, $\mathrm{H}_\nu^{(1)}$, we will give a short digression on Hankel and Bessel functions, as well as the necessary asymptotic expansion.

For the rest of this section, Γ denotes the usual Γ-function and not the matrix of fundamental solutions of the Stokes (resolvent) problem. Moreover, for this whole section we assume that $z \in \mathbb{C} \setminus (-\infty, 0]$, so that all powers of complex numbers and the complex logarithm are well-defined. Note that $\mathrm{H}_\nu^{(1)}$ is defined as $\mathrm{J}_\nu + \mathrm{i}\mathrm{Y}_\nu$, see Lebedev [63, Eq. (5.6.1)], and that (4.9) is a representation formula of this function. Here J_ν is the *Bessel function of the first kind of order* ν and Y_ν is the *Bessel function of the second kind of order* ν. For $\nu \in \mathbb{R}$, these functions are given by

$$\mathrm{J}_\nu(z) := \sum_{l=0}^\infty \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+\nu+1)}\Big(\frac{z}{2}\Big)^{\nu+2l},$$

see [63, Sec. 5.3], and

$$\begin{aligned} \mathrm{Y}_\nu(z) &:= \frac{\mathrm{J}_\nu(z)\cos(\nu\pi)-\mathrm{J}_{-\nu}(z)}{\sin(\nu\pi)} \qquad (\nu \notin \mathbb{Z}) \\ \mathrm{Y}_n(z) &:= \lim_{\nu\to n} \mathrm{Y}_\nu(z) \qquad (n \in \mathbb{Z}), \end{aligned} \tag{4.17}$$

see [63, Sec. 5.4]. The discussion in [63, Sec. 5.4] shows that the limit above always exists. For non-negative integers n, one can also determine a representation of Y_n as a power series, see [63, Eq. (5.5.1)], namely,

$$\begin{aligned} \mathrm{Y}_n(z) = &-\frac{1}{\pi}\sum_{l=0}^{n-1}\frac{(n-l-1)!}{l!}\Big(\frac{z}{2}\Big)^{2l-n} \\ &+\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!(n+l)!}\Big(\frac{z}{2}\Big)^{n+2l}\Big[2\log\Big(\frac{z}{2}\Big)-\psi(l+1)-\psi(l+n+1)\Big], \end{aligned}$$

where

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

Moreover, for Bessel functions of the first kind and half-integer order, one finds in [63, Sec. 5.8] that for $n \in \mathbb{N}_0$

$$\begin{aligned} \mathrm{J}_{n+\frac{1}{2}}(z) &= (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} z^{n+\frac{1}{2}} \left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^n \frac{\sin(z)}{z} \\ \mathrm{J}_{-\frac{1}{2}-n}(z) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} z^{n+\frac{1}{2}} \left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^{n+1} \sin(z). \end{aligned} \tag{4.18}$$

Using that $\mathrm{H}_n^{(1)} = \mathrm{J}_n + \mathrm{i}\mathrm{Y}_n$ and the representation formulas above, one obtains that the Hankel functions of the first kind, of non-negative integer order are given by

$$\begin{aligned} z^n \mathrm{H}_n^{(1)}(z) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(l+n)!2^{n+2l}} z^{2n+2l} - \frac{\mathrm{i}}{\pi} \sum_{l=0}^{n-1} \frac{(n-l-1)!}{l!2^{2l-n}} z^{2l} \\ &+ \frac{\mathrm{i}}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!2^{n+2l}} z^{2n+2l} \Big[2\log\left(\frac{z}{2}\right) - \psi(l+1) - \psi(l+n+1) \Big]. \end{aligned}$$

Calculating the first few terms, we find

$$\begin{aligned} z\mathrm{H}_1^{(1)}(z) &= \frac{2^1\Gamma(1)}{\pi \mathrm{i}} + \frac{(2\gamma-1)\mathrm{i}+\pi}{2\pi} z^2 + \frac{\mathrm{i}}{\pi} z^2 \log\left(\frac{z}{2}\right) \\ &\quad + z^4 \Big[g_1(z) \log\left(\frac{z}{2}\right) + h_1(z) \Big] \\ z^2\mathrm{H}_2^{(1)}(z) &= \frac{2^2\Gamma(2)}{\pi \mathrm{i}} + \frac{\Gamma(2-1)}{4\pi \mathrm{i}} z^2 + z^4 \Big[g_2(z) \log\left(\frac{z}{2}\right) + h_2(z) \Big] \\ z^n\mathrm{H}_n^{(1)}(z) &= \frac{2^n\Gamma(n)}{\pi \mathrm{i}} + \frac{2^n\Gamma(n-1)}{4\pi \mathrm{i}} z^2 + z^4 h_n(z) + z^{2n} g_n(z) \log\left(\frac{z}{2}\right), \end{aligned} \tag{4.19}$$

where $\gamma = -\Psi(1)$ denotes the Euler-Masceroni constant, $n = 3, 4, \dots$, and g_k, h_k are entire functions for all $k \in \mathbb{N}$. Appealing to (4.17), one immediately sees that in the half-integer case the identity $\mathrm{Y}_{n+\frac{1}{2}} = (-1)^{n+1}\mathrm{J}_{-(n+\frac{1}{2})}$ holds. This, together with $\mathrm{H}_\nu^{(1)} = \mathrm{J}_\nu + \mathrm{i}\mathrm{Y}_\nu$ and (4.18) yields

$$\begin{aligned} & z^{n+\frac{1}{2}} \mathrm{H}_{n+\frac{1}{2}}^{(1)}(z) \\ &= (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Big[z^{2n+1} \left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^n \frac{\sin(z)}{z} - \mathrm{i} z^{2n+1} \left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^{n+1} \sin(z) \Big]. \end{aligned}$$

To obtain an asymptotic expansion, expand the sine-function into a power series. This leads to the appearance of terms of the form $(\frac{\mathrm{d}}{z\mathrm{d}z})^n z^j$ for $j \in \mathbb{N}_0$. Calculating these expressions iteratively yields

$$\left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^n z^j = \begin{cases} \prod_{l=0}^{n-1}(j-2l)z^{j-2n}, & j \text{ odd or } j \geq 2n \\ 0, & j \text{ even and } j < 2n. \end{cases}$$

With this, we derive

$$\begin{aligned} z^{n+\frac{1}{2}}\mathrm{H}^{(1)}_{n+\frac{1}{2}}(z) &= (-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big[z^{2n+1}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^n z^{2j} \\ &\quad - \mathrm{i}z^{2n+1}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^{n+1} z^{2j+1}\Big] \\ &= (-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big[\sum_{j=n}^{\infty}\frac{(-1)^j}{(2j+1)!}\prod_{l=0}^{n-1}(2j-2l)z^{2j+1} \\ &\quad - \mathrm{i}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\prod_{l=0}^{n}(2j+1-2l)z^{2j}\Big]. \end{aligned}$$

Next, use

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \qquad \Gamma(n+\tfrac{1}{2}) = \frac{\prod_{l=1}^{n}(2l-1)}{2^n}\sqrt{\pi},$$

where $n \in \mathbb{N}$, see [63, Eq. (1.2.5), Eq. (1.2.6)], to conclude the validity of the identities

$$(-1)^n\frac{\sqrt{2}}{\sqrt{\pi}}(-\mathrm{i})\prod_{l=1}^{n}(1-2l) = \frac{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}{\pi\mathrm{i}}$$

and

$$(-1)^n\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\mathrm{i}}{6}\prod_{l=0}^{n}(3-2l) = \frac{2^{n+\frac{1}{2}}\Gamma(n-1+\frac{1}{2})}{4\pi\mathrm{i}}.$$

By these means, calculate the first few terms of $z^{n+\frac{1}{2}}\mathrm{H}^{(1)}_{n+\frac{1}{2}}(z)$ for $n \geq 1$ as

$$\begin{aligned} z^{\frac{3}{2}}\mathrm{H}^{(1)}_{\frac{3}{2}}(z) &= \frac{\sqrt{2}}{\sqrt{\pi}\mathrm{i}} + \frac{1}{\sqrt{2\pi}\mathrm{i}}z^2 + \frac{\sqrt{2}}{3\sqrt{\pi}}z^3 + z^4\widetilde{g}_1(z) \\ &= \frac{2^{\frac{3}{2}}\Gamma(\frac{3}{2})}{\pi\mathrm{i}} + \frac{2^{\frac{3}{2}}\Gamma(\frac{3}{2}-1)}{4\pi\mathrm{i}}z^2 + \frac{\sqrt{2}}{3\sqrt{\pi}}z^3 + z^4\widetilde{g}_1(z) \end{aligned} \tag{4.20}$$

$$z^{n+\frac{1}{2}}\mathrm{H}^{(1)}_{n+\frac{1}{2}}(z) = \frac{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}{\pi\mathrm{i}} + \frac{2^{n+\frac{1}{2}}\Gamma(n-1+\frac{1}{2})}{4\pi\mathrm{i}}z^2 + z^4\widetilde{g}_n(z), \tag{4.21}$$

for $n = 2, 3, \dots$ and entire functions $\widetilde{g}_1$ and $\widetilde{g}_n$.

4.3 Regularity theory for the L^2-Dirichlet problem of the Stokes resolvent

In this section, we will establish the regularity theory to the L^2-Dirichlet problem of the Stokes resolvent. In other words, we will show that ($\mathrm{Reg}_{\mathrm{S},\lambda}$) is uniquely solvable. As uniqueness directly follows by Theorem 4.1.7 the solvability of ($\mathrm{Reg}_{\mathrm{S},\lambda}$) shows that a solution of the L^2-Dirichlet problem with boundary data in $\mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ enjoys additional non-tangential estimates on its gradient and the corresponding pressure.

Considering (4.8), the definition of the number k in (4.7), (4.11), as well as the asymptotic expansions (4.20) and (4.21) it is immediate that the matrix of fundamental solutions of the Stokes resolvent problem is smooth in $\mathbb{R}^d \setminus \{0\}$. Thus, the single layer potential $\mathcal{S}_\lambda f$ is smooth in $\mathbb{R}^d \setminus \partial\Omega$ and one differentiates $\mathcal{S}_\lambda f$ by differentiating the integrand. This shows that $u := \mathcal{S}_\lambda f$ and ϕ, as it is defined in (4.6), solve (SRP) in $\mathbb{R}^d \setminus \partial\Omega$.

Recall the non-tangential maximal function N_a defined in (4.13). The single layer potential was thoroughly investigated by Shen in [89, Sec. 3]. For example, in [89, Lem. 3.2, Lem. 3.3] it is shown that $N_a \mathcal{S}_\lambda f$ and $N_a \nabla \mathcal{S}_\lambda f$ lie in $\mathrm{L}^2(\partial\Omega)$ for some $a > 0$ whenever f lies in $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. Additionally, it is shown that the non-tangential limits of $\partial_j \mathcal{S}_\lambda f$ taken from inside and outside Ω exist. By means of these facts, the following proposition proves the existence of the non-tangential limits of $\mathcal{S}_\lambda f$.

Proposition 4.3.1. *Let $\theta \in [0, \pi)$ and $\lambda \in \mathrm{S}_\theta$. Then, there exists a constant $C > 0$ depending only on d, θ, and the Lipschitz character of Ω such that for all $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$*

$$\Big\| \int_{\partial\Omega} |\Gamma(\cdot - y; \lambda)| \, |f(y)| \, \mathrm{d}\sigma(y) \Big\|_{\mathrm{L}^2(\partial\Omega)} \leq C \|f\|_{\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)}.$$

Moreover, $\mathcal{S}_\lambda f$ converges to

$$\partial\Omega \ni q \mapsto \int_{\partial\Omega} \Gamma(q - y; \lambda) f(y) \, \mathrm{d}\sigma(y)$$

non-tangentially σ-a.e. from inside and outside Ω.

One cornerstone of the proof of the proposition is the following lemma, which is stated separately for further use.

Lemma 4.3.2. *Let $x \in \mathbb{R}^d$, $0 < \varepsilon \leq r_0/4$, and $l \in \mathbb{N}_0$ with $l < d-1$. Then there exists a constant $C > 0$ depending only on d, l, and M such that*

$$\int_{\partial\Omega \cap B(x,\varepsilon)} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y) \leq C\varepsilon^{d-l-1}.$$

Proof. Decompose the domain of integration into annuli

$$\int_{\partial\Omega \cap B(x,\varepsilon)} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y) \leq \sum_{n=-\infty}^{n_0} \int_{\partial\Omega \cap [B(x,2^{n+1}) \setminus B(x,2^n)]} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y),$$

where n_0 is subject to the condition $2^{n_0} < \varepsilon \leq 2^{n_0+1}$. Note $2^{n_0+2} \leq r_0$ by the assumption on ε. Estimate the integrand by means of the inner radii and the measure of the respective sets by means of the outer radii to deduce

$$\leq \sum_{n=-\infty}^{n_0} 2^{-nl} \sigma(\partial\Omega \cap B(x, 2^{n+1})).$$

Next, if $\partial\Omega \cap B(x,2^n) \neq \emptyset$, there exists $q_x \in \partial\Omega$ with $B(x,2^{n+1}) \subset B(q_x, 2^{n+2})$. Since $2^{n_0+2} \leq r_0$, we can appeal to Lemma 4.1.14 to deduce that there exists a constant $C > 0$ depending only on d and M such that

$$\sigma(\partial\Omega \cap B(x,2^{n+1})) \leq \sigma(\partial\Omega \cap B(q_x, 2^{n+2})) \leq C 2^{(d-1)(n+2)}.$$

We conclude that

$$\int_{\partial\Omega \cap B(x,\varepsilon)} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y) \leq C \sum_{n=-\infty}^{n_0} 2^{-nl+(d-1)(n+2)}.$$

Calculating the exponent first and performing then an index shift in the series yields

$$= C 2^{2d-2} 2^{n_0(d-l-1)} \sum_{n=-\infty}^{0} 2^{n(d-l-1)}.$$

Since $2^{n_0} \leq \varepsilon$ and $d-l-1 > 0$, we conclude the proof. $\square$

Proof of Proposition 4.3.1. Let $q \in \partial\Omega$ and begin the proof by estimating $\|\Gamma(q - \cdot\,; \lambda)\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})}$ as follows. An application of [89, Thm. 2.4] yields a constant $C > 0$ depending only on d and θ such that

$$\int_{\partial\Omega} |\Gamma(q - y; \lambda)| \; \mathrm{d}\sigma(y) \leq C \int_{\partial\Omega} \frac{1}{|q - y|^{d-2}} \; \mathrm{d}\sigma(y).$$

Next, decompose the domain of integration as

$$\begin{aligned} &\leq C \int_{\partial\Omega \cap B(q,r_0/4)} \frac{1}{|q - y|^{d-2}} \; \mathrm{d}\sigma(y) \\ &\quad + C \int_{\partial\Omega \setminus B(q,r_0/4)} \frac{1}{|q - y|^{d-2}} \; \mathrm{d}\sigma(y). \end{aligned}$$

Apply Lemma 4.3.2 to the first term on the right-hand side and estimate the second term on the right-hand side by means of the inner radius to deduce

$$\leq C r_0 + C 4^{d-2} r_0^{2-d} \sigma(\partial\Omega),$$

where C depends only on d, θ, and M. Finally, invoke Lemma 4.1.14 to bound $\sigma(\partial\Omega)$, so that the right-hand side is less than a constant $\mathcal{C} > 0$ depending only on d, θ, r_0, and the Lipschitz character of Ω.

Due to the smoothness of the fundamental solution away of the origin, Proposition 1.1.4 is applicable and yields together with the radial symmetry of $\Gamma(x - y; \lambda)$

$$\Big\| \int_{\partial\Omega} |\Gamma(\cdot - y; \lambda)| \, |f(y)| \; \mathrm{d}\sigma(y) \Big\|_{\mathrm{L}^2(\partial\Omega)} \leq \mathcal{C} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \tag{4.22}$$

By virtue of [89, Lem. 3.2] together with Proposition 4.1.11, $N_a[\nabla \mathcal{S}_\lambda f]$ lies in $\mathrm{L}^2(\partial\Omega)$ for every $a > 0$. Thus, if $\{\Gamma(q)\}_{q\in\partial\Omega}$ is any regular family of cones, which lies either inside or outside Ω, an application of Lemma 4.1.12 reveals

$$(\nabla \mathcal{S}_\lambda f)^*(q) < \infty \qquad (\sigma\text{-a.e. } q \in \partial\Omega).$$

For the rest of this proof fix q such that $\int_{\partial\Omega} |\Gamma(q - y; \lambda)| \, |f(y)| \; \mathrm{d}\sigma(y)$ and $(\nabla \mathcal{S}_\lambda f)^*(q)$ are finite. Let $(x_k)_{k\in\mathbb{N}} \subset \Gamma(q)$ with $x_k \to q$. Because $\Gamma(q)$ is convex, all convex combinations of x_k and x_m lie inside $\Gamma(q)$. Combining this fact with the fundamental theorem of calculus shows for the jth component of $\mathcal{S}_\lambda f$

$$\begin{aligned}
&|([\mathcal{S}_\lambda f](x_k))_j - ([\mathcal{S}_\lambda f](x_m))_j| \\
&\quad = \Big| \int_0^1 \Big\langle (\nabla[\mathcal{S}_\lambda f](tx_k + (1-t)x_m))_j, (x_k - x_m) \Big\rangle \,\mathrm{d}t \Big| \\
&\quad \leq |(\nabla \mathcal{S}_\lambda f)^*(q)|\,|x_k - x_m|\,.
\end{aligned} \tag{4.23}$$

We conclude that $\mathcal{S}_\lambda f$ converges non-tangentially σ-a.e. To identify the limit, we proceed by an approximation argument. Note that $\mathrm{C}(\partial\Omega; \mathbb{C}^d)$ is dense in $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ by Rudin [83, Thm. 3.14] combined with Remark 1.3.6. Let $0 < \varepsilon < r_0/8$ and $f \in \mathrm{C}(\partial\Omega; \mathbb{C}^d)$, then

$$\begin{aligned}
&\Big| \int_{\partial\Omega} [\Gamma(x_k - y; \lambda) - \Gamma(q - y; \lambda)] f(y) \,\mathrm{d}\sigma(y) \Big| \\
&\quad \leq \int_{\partial\Omega \setminus B(q,\varepsilon)} |\Gamma(x_k - y; \lambda) - \Gamma(q - y; \lambda)|\,|f(y)| \,\mathrm{d}\sigma(y) \|f\|_{\mathrm{L}^\infty(\partial\Omega;\mathbb{C}^d)} \\
&\qquad + \int_{\partial\Omega \cap B(q,\varepsilon)} |\Gamma(x_k - y; \lambda)| + |\Gamma(q - y; \lambda)| \,\mathrm{d}\sigma(y) \|f\|_{\mathrm{L}^\infty(\partial\Omega;\mathbb{C}^d)}.
\end{aligned}$$

Clearly, the first integral on the right-hand side converges to zero by smoothness of $\Gamma(\cdot; \lambda)$ on $\mathbb{R}^d \setminus \{0\}$.

Next, by [89, Thm. 2.4] there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_{\partial\Omega \cap B(q,\varepsilon)} |\Gamma(x_k - y; \lambda)| + |\Gamma(q - y; \lambda)| \,\mathrm{d}\sigma(y) \\
&\quad \leq C \int_{\partial\Omega \cap B(q,\varepsilon)} \frac{1}{|x_k - y|^{d-2}} + \frac{1}{|q - y|^{d-2}} \,\mathrm{d}\sigma(y) \\
&\quad \leq C \int_{\partial\Omega \cap B(x_k, \varepsilon + |x_k - q|)} \frac{1}{|x_k - y|^{d-2}} \,\mathrm{d}\sigma(y) \\
&\qquad + C \int_{\partial\Omega \cap B(q,\varepsilon)} \frac{1}{|q - y|^{d-2}} \,\mathrm{d}\sigma(y).
\end{aligned}$$

Consequently, by virtue of Lemma 4.3.2, we find

$$\limsup_{k\to\infty} \int_{\partial\Omega \cap B(q,\varepsilon)} [|\Gamma(x_k - y; \lambda)| + |\Gamma(q - y; \lambda)|] \,\mathrm{d}\sigma(y) \leq C\varepsilon.$$

It follows that the limit of $([\mathcal{S}_\lambda f](x_k))_{k\in\mathbb{N}}$ is

$$\int_{\partial\Omega} \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y), \tag{4.24}$$

whenever $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$. Next, Shen proved in [89, Lem. 3.2] that there exists a constant $C > 0$ such that

$$\|N_a \mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega)} \leq C \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)),$$

so that by Proposition 4.1.11 and Lemma 4.1.12 the same holds true for $N_a \mathcal{S}_\lambda f$ replaced by $(\mathcal{S}_\lambda f)^*$.

To conclude the proof, define an operator $\widetilde{\mathcal{S}}_\lambda : \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, which acts on f as the non-tangential limit of $\mathcal{S}_\lambda f$ at the boundary. As the non-tangential limit at $q \in \partial\Omega$ is always less than $([\mathcal{S}_\lambda f])^*(q)$ (whenever the limit exists), the operator $\widetilde{\mathcal{S}}_\lambda$ is bounded on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$. Moreover, $\widetilde{\mathcal{S}}_\lambda$ coincides with the bounded operator

$$\mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \ni f \mapsto \Big[\partial\Omega \ni q \mapsto \int_{\partial\Omega} \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\Big]$$

on $\mathrm{C}(\partial\Omega;\mathbb{C}^d)$. Consequently, by density they have to coincide on all of $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ so that $[\mathcal{S}_\lambda f](q)$ converges to (4.24) for σ-almost every $q \in \partial\Omega$ and every $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$. □

The preceding lemma suggests to interpret the single layer potential $\mathcal{S}_\lambda$ twofold. The first interpretation is that $\mathcal{S}_\lambda$ produces solutions to the Stokes resolvent problem (SRP) in Ω from functions $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$. The second is that $\mathcal{S}_\lambda$ gives rise to a bounded operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, which realizes the non-tangential limit of $\mathcal{S}_\lambda f$ to $\partial\Omega$. In the following, we will use this notation in both meanings.

As an operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, $\mathcal{S}_\lambda$ has the following properties.

Proposition 4.3.3. *Let* $\theta \in [0,\pi)$ *and* $\lambda \in \mathrm{S}_\theta$. *Then the operator* $\mathcal{S}_\lambda$ *is a bounded operator from* $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ *into* $\mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$ *whose restriction onto* $\mathrm{L}^2_\nu(\partial\Omega)$ *is injective. The operator norm depends only on* d, θ, r_0, *and the Lipschitz character of* Ω.

Furthermore, for all $\tau \in (0, \sigma(\partial\Omega)^{-2})$ *there exists a constant* $C > 0$ *depending only on* d, θ, τ, r_0, *and the Lipschitz character of* Ω, *such that* *for all* $f \in \mathrm{L}^2_\nu(\partial\Omega)$ *and all* $|\lambda| \geq \tau$

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C\Big\{\|\nabla_{\tan}\mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} + |\lambda|^{\frac{1}{2}}\,\|\mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} + |\lambda|\,\|\langle\nu,\mathcal{S}_\lambda f\rangle\|_{\mathrm{W}^{1,2}(\partial\Omega)^*}\Big\}.$$

Proof. The boundedness of $\mathcal{S}_\lambda$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ was proven in Proposition 4.3.1. To prove $\mathcal{S}_\lambda f \in \mathrm{L}^2_\nu(\partial\Omega)$, approximate Ω by smooth domains Ω_k from inside by means of Proposition 1.3.19. In Ω_k we have $\operatorname{div}(\mathcal{S}_\lambda f) = 0$ so that by the divergence theorem

$$\int_{\partial\Omega_k} \langle\mathcal{S}_\lambda f, \nu_k\rangle \,\mathrm{d}\sigma_k = 0,$$

where σ_k denotes the surface measure on $\partial\Omega_k$ and ν_k the outward unit normal to $\partial\Omega_k$. Proceeding as in the proof of Lemma 1.3.21 shows that the non-tangential convergence of $\mathcal{S}_\lambda f$ and the fact $(\mathcal{S}_\lambda f)^* \in \mathrm{L}^2(\partial\Omega)$, cf. [89, Lem. 3.2] together with Proposition 4.1.11 and Lemma 4.1.12, imply that this equality holds true in the limit.

Moreover, $\mathcal{S}_\lambda f \in \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$ follows by Lemma 1.3.21, since, additionally to the facts above, $\nabla\mathcal{S}_\lambda f$ converges non-tangentially σ-a.e. and $(\nabla\mathcal{S}_\lambda f)^* \in \mathrm{L}^2(\partial\Omega)$, cf. [89, Lem. 3.2, Lem. 3.3] together with Proposition 4.1.11 and Lemma 4.1.12. Combining the estimates provided by these results delivers

$$\|\nabla_{\tan}\mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} \leq 2\|(\nabla\mathcal{S}_\lambda f)^*\|_{\mathrm{L}^2(\partial\Omega)} \leq 2C\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)},$$

where C depends solely on d, θ, and the Lipschitz character of Ω. It follows that $\mathcal{S}_\lambda : \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$ is bounded.

To prove the injectivity, we rely on ideas of the proof of [89, Lem. 5.1]. Note that $\mathcal{S}_\lambda f$ has the following scaling behavior

$$[\mathcal{S}_\lambda f](rq) = [\mathcal{S}_{r^2\lambda} r f(r\cdot)](q) \qquad (q \in r^{-1}\partial\Omega). \tag{4.25}$$

This follows simply by a change of variables and by (4.12). Thus, without loss of generality, we can assume that $\sigma(\partial\Omega) = 1$.

Let $f \in \mathrm{L}^2_\nu(\partial\Omega)$ be such that $\mathcal{S}_\lambda f = 0$ and recall the pressure ϕ corresponding to $\mathcal{S}_\lambda f$ defined by (4.6). By virtue of [89, Lem. 3.3] the non-tangential limits of $\mathcal{S}_\lambda f$ and of ϕ from outside Ω exist. Recall that non-tangential limits taken from inside Ω are denoted by the subscript $+$ and that non-tangential limits taken outside Ω are denoted by the subscript $-$, see Remark 1.3.18 (2). Define the expression

$$[\partial_\nu(\mathcal{S}_\lambda f, \phi)]_- := \begin{pmatrix} \langle \nu, [\nabla(\mathcal{S}_\lambda f)_1]_- \rangle \\ \vdots \\ \langle \nu, [\nabla(\mathcal{S}_\lambda f)_d]_- \rangle \end{pmatrix} - \phi_- \nu.$$

Here, $(\mathcal{S}_\lambda f)_j$ denotes the jth component of $\mathcal{S}_\lambda f$ and ν denotes the exterior unit normal to $\partial\Omega$. With this quantity, the inequality

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C \|[\partial_\nu(\mathcal{S}_\lambda f, \phi)]_-\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}$$

was established in [89, Lem. 5.2], with a constant $C > 0$ depending only on d, θ, and the Lipschitz character of Ω. Note that the assumptions $f \in \mathrm{L}^2_\nu(\partial\Omega)$ and $\sigma(\partial\Omega) = 1$ were used in this situation. By the triangle and the Cauchy–Schwarz inequality, we find

$$\leq C \Big\{ \|[\nabla \mathcal{S}_\lambda f]_-\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} + \|\phi_-\|_{\mathrm{L}^2(\partial\Omega)} \Big\}.$$

For $|\lambda| \geq \tau \in (0,1)$, an application of [89, Thm. 4.6] shows, that this is controlled by

$$\begin{aligned} \leq C \Big\{ &\|\nabla_{\tan}[\mathcal{S}_\lambda f]_-\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} + |\lambda|^{\frac{1}{2}} \|[\mathcal{S}_\lambda f]_-\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \\ &+ |\lambda| \, \|\langle \nu, [\mathcal{S}_\lambda f]_- \rangle\|_{\mathrm{W}^{1,2}(\partial\Omega)^*} \Big\}, \end{aligned}$$

with a different constant C, depending only on d, θ, τ, and the Lipschitz character of Ω. By Proposition 4.3.1, we find $[\mathcal{S}_\lambda f]_- = [\mathcal{S}_\lambda f]_+$ and by assumption, we know that $[\mathcal{S}_\lambda f]_+ = 0$. It follows that $f = 0$ and that the restriction of $\mathcal{S}_\lambda$ to $\mathrm{L}^2_\nu(\partial\Omega)$ is injective.

Note that the desired estimate stated in the proposition follows from the calculation above by rescaling, cf. (4.25). $\square$

The paragraph before Theorem 4.15 in the work of Fabes, Kenig, and Verchota [28] shows that $\mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$ is invertible and hence a Fredholm operator. The following two lemmas show that $\mathcal{S}_\lambda - \mathcal{S}_0$ is compact and so, that $\mathcal{S}_\lambda : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$

is a Fredholm operator as well. The first lemma is needed in order to prove the required kernel estimates for the compactness and bases on the results of Section 4.2.

Even though Shen investigated the function $|\Gamma(x;\lambda)-\Gamma(x;0)|$ in [89, Sec. 2], he needed only estimates on its gradient. In consequence, he did not state estimates on the function $|\Gamma(x;\lambda)-\Gamma(x;0)|$ itself. This is the reason why we include the following proof.

Lemma 4.3.4. *Let $\theta \in [0,\pi)$, $\lambda \in \mathrm{S}_\theta$, and $x \in \mathbb{R}^d \setminus \{0\}$. If $|\lambda|\,|x|^2 \leq \mathrm{e}^{-2}$, there exists a constant $C>0$ depending only on θ and d such that*

$$|\Gamma(x;\lambda)-\Gamma(x;0)| \leq C \begin{cases} |\lambda|^{1/2}, & \text{if } d=3 \\ |\lambda|\,|x|^{4-d}\,|\log(|\lambda|\,|x|^2)|, & \text{if } d=4,6 \\ |\lambda|\,|x|^{4-d}, & \text{if } d=5, d\geq 7. \end{cases}$$

Proof. This proof closely follows the proof of [89, Thm. 2.5] and uses several identities established there. Recall the definition of k in (4.7).

Considering the first lines of the proof of [89, Thm. 2.5], we find that in the case $d \neq 4$ the following identity is valid

$$\begin{aligned} &\Gamma_{\alpha\beta}(x;\lambda)-\Gamma_{\alpha\beta}(x;0) \\ &\quad = \{\mathrm{G}(x;\lambda)-\mathrm{G}(x;0)\}\delta_{\alpha\beta} \\ &\qquad - \frac{1}{\lambda}\partial_\alpha\partial_\beta\Big\{\mathrm{G}(x;\lambda)-\mathrm{G}(x;0)+\frac{\lambda}{2\omega_d(d-2)(d-4)\,|x|^{d-4}}\Big\}. \end{aligned} \tag{4.26}$$

Next, in the case $d \geq 5$ by [89, Lem. 2.2], and in the cases $d=3,4$ by [89, Rem. 2.3], there exists a constant $C>0$ depending only on d and θ such that

$$|\mathrm{G}(x;\lambda)-\mathrm{G}(x;0)| \leq C \begin{cases} |\lambda|^{1/2}, & \text{if } d=3 \\ |\lambda|\,(|\log(|\lambda|\,|x|^2)|+1), & \text{if } d=4 \\ |\lambda|\,|x|^{4-d}, & \text{if } d\geq 5. \end{cases} \tag{4.27}$$

Note, in order to derive (4.27), it is required that $|\lambda|\,|x|^2 \leq 1/2$, which is fulfilled by the assumptions. This already proves the desired estimate for the first term on the right-hand side of (4.26) in the case $d \neq 4$. It remains to consider the second term on the right-hand side of (4.26) and the case

$d = 4$. Here, we first concentrate on the case $d \geq 5$. In this situation, the following identity was established in [89, Eq. (2.28)]

$$\begin{aligned}\mathrm{G}(x;\lambda) - \mathrm{G}(x;0) + \frac{\lambda}{2\omega_d(d-2)(d-4)\,|x|^{d-4}} \\ = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{1}{|x|^{d-2}} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\},\end{aligned} \tag{4.28}$$

where $z := k\,|x|$,

$$a_d := \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1)}{\pi \mathrm{i}}, \quad \text{and} \quad b_d := -\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-2)(d-4)} = \frac{\mathrm{i}2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-2)}{4\pi\mathrm{i}}.$$

Denote the right-hand side of (4.28) by $\mathcal{G}(x)$. Then, its first derivatives are given by means of the chain and product rule by

$$\begin{aligned}\partial_\beta \mathcal{G}(x) &= \frac{(2-d)\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{x_\beta}{|x|^d} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \\ &\quad + \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{k x_\beta}{|x|^{d-1}} \frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\}.\end{aligned}$$

Calculating the second derivatives yields

$$\begin{aligned}&\partial_\alpha \partial_\beta \mathcal{G}(x) \\ &= \frac{(2-d)\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{\delta_{\alpha\beta}\,|x|^2 - d x_\alpha x_\beta}{|x|^{d+2}} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \\ &\quad + \frac{(2-d)\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{k x_\alpha x_\beta}{|x|^{d+1}} \frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \\ &\quad + \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{k(\delta_{\alpha\beta}\,|x|^2 - (d-1) x_\alpha x_\beta)}{|x|^{d+1}} \frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \\ &\quad + \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \frac{k^2 x_\alpha x_\beta}{|x|^d} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\}.\end{aligned}$$

In the case $d \geq 7$, Shen proved in [89, Eq. (2.29)] that there exists a constant $C > 0$ depending only on d and θ such that for all $0 \leq l \leq 2$ and $z \in \mathbb{C}$ with $|z| < 1/2$ and $\operatorname{Im}(z) > 0$

$$\Big| \frac{\mathrm{d}^l}{\mathrm{d}z^l} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \Big| \leq C\,|z|^{4-l}.$$

Thus, for $d \geq 7$, we obtain by virtue of (4.7)

$$
\begin{aligned}
|\partial_\alpha \partial_\beta \mathcal{G}(x)| &\leq C\Big\{ |x|^{-d} \, |k\,|x||^4 + |k|\, |x|^{1-d} \, |k\,|x||^3 + |k|^2 \, |x|^{2-d} \, |k\,|x||^2 \Big\} \\
&= C\, |\lambda|^2 \, |x|^{4-d} .
\end{aligned}
$$

If $d = 6$, [89, Eq. (2.31)] reads as

$$
\left| \frac{\mathrm{d}^l}{\mathrm{d}z^l} \Big\{ z^{\frac{d}{2}-1} \mathrm{H}^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \right| \leq C\, |z|^{4-l} \, |\log(z)| ,
$$

which holds for all $0 \leq l \leq 2$ and $z \in \mathbb{C}$ with $|z| < 1/2$ and $\mathrm{Im}(z) > 0$. Thus, similarly as above, we derive

$$
|\partial_\alpha \partial_\beta \mathcal{G}(x)| \leq C\, |\lambda|^2 \, |x|^{4-d} \, |\log(k\,|x|)| .
$$

Moreover, because

$$
|\log(z)| = |\log(|z|) + \mathrm{i} \arg(z)| \leq (1+\pi)\, |\log(|z|)| \qquad (0 < |z| \leq \mathrm{e}^{-1})
$$

one shows together with $2\,|\log(|z|)| = |\log(|z|^2)|$ that

$$
|\partial_\alpha \partial_\beta \mathcal{G}(x)| \leq C\, |\lambda|^2 \, |x|^{4-d} \, |\log(|\lambda|\, |x|^2)|,
$$

thereby establishing the desired estimate in the case $d = 6$. If $d = 5$, one proceeds as Shen below [89, Eq. (2.32)] and writes

$$
\partial_\alpha \partial_\beta \mathcal{G}(x) = \partial_\alpha \partial_\beta \left\{ \frac{\mathrm{i}}{4(2\pi)^{3/2}} \frac{1}{|x|^3} \Big\{ z^{\frac{3}{2}} \mathrm{H}^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - w z^3 \Big\} \right\}
$$

where $z = k\,|x|$, $w \in \mathbb{C}$ is arbitrary, and a_5 and b_5 are given below (4.28). Note that w can be chosen arbitrarily, because $z^3/\,|x|^3$ is constant. By virtue of (4.20), fix the value of w as $\sqrt{2}/(3\sqrt{\pi})$ and use the product rule to get

$$
\begin{aligned}
|\partial_\alpha \partial_\beta \mathcal{G}(x)| \leq C\Bigg\{ & \frac{1}{|x|^5} \Big| z^{\frac{3}{2}} \mathrm{H}^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - w z^3 \Big| \\
&+ \frac{|k|}{|x|^4} \left| \frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z^{\frac{3}{2}} \mathrm{H}^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - w z^3 \Big\} \right| \\
&+ \frac{|k|^2}{|x|^3} \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \Big\{ z^{\frac{3}{2}} \mathrm{H}^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - w z^3 \Big\} \right| \Bigg\}.
\end{aligned}
$$

Now, (4.20) shows that there exists a constant $C > 0$ depending only on d such that

$$\leq C \frac{|\lambda|^2}{|x|}.$$

In the case $d = 3$ one uses the explicit expressions (4.2) and (4.10) to derive

$$\begin{aligned}\mathcal{G}(x) = \mathrm{G}(x;\lambda) - \mathrm{G}(x;0) - \frac{\lambda\,|x|}{8\pi} &= \frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{4\pi\,|x|} - \frac{\lambda\,|x|}{8\pi} \\ &= \frac{1}{4\pi\,|x|}\sum_{n=1}^{\infty}\frac{(\mathrm{i}k\,|x|)^n}{n!} - \frac{\lambda\,|x|}{8\pi} \\ &= \frac{\mathrm{i}k}{4\pi} + \frac{1}{4\pi}\sum_{n=3}^{\infty}\frac{(\mathrm{i}k)^n\,|x|^{n-1}}{n!}.\end{aligned}$$

Thus,

$$|\partial_\alpha\partial_\beta\mathcal{G}(x)| \leq \frac{1}{4\pi}\sum_{n=3}^{\infty}\frac{|\lambda|^{\frac{n}{2}}\,(n-1)(n-2)\,|x|^{n-3}}{n!} \leq \frac{|\lambda|^{\frac{3}{2}}}{4\pi}\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{\mathrm{e}^{n-3}n!},$$

where we used that $|\lambda|^{\frac{1}{2}}\,|x| \leq 1/\mathrm{e}$. This establishes the desired estimate in the case $d = 3$. If $d = 4$ one can write, cf. [89, p. 404],

$$\begin{aligned}&\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0) \\ &\qquad = \{\mathrm{G}(x;\lambda) - \mathrm{G}(x;0)\}\delta_{\alpha\beta} \\ &\qquad\quad - \frac{\mathrm{i}}{\lambda}\partial_\alpha\partial_\beta\bigg\{\frac{1}{8\pi\,|x|^2}\Big\{z\mathrm{H}_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2\log(z)\Big\}\bigg\},\end{aligned}$$

where z, a_4, and b_4 are as above and where w is an arbitrary complex number. Define

$$\mathcal{G}(x) := \frac{1}{8\pi\,|x|^2}\Big\{z\mathrm{H}_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2\log(z)\Big\}$$

and choose $w := [(2\gamma - 1 - 2\log(2))\mathrm{i} + \pi]/(2\pi)$, where γ denotes the Euler-Masceroni constant. Using the chain and the product rule, we deduce

$$
\begin{aligned}
|\partial_\alpha \partial_\beta \mathcal{G}(x)| \leq C \bigg\{ & \frac{1}{|x|^4} \left| z\mathrm{H}_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2 \log(z) \right| \\
& + \frac{|k|}{|x|^3} \left| \frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z\mathrm{H}_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2 \log(z) \Big\} \right| \\
& + \frac{|k|^2}{|x|^2} \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \Big\{ z\mathrm{H}_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2 \log(z) \Big\} \right| \bigg\}.
\end{aligned}
$$

Now, by (4.19) together with $\log(z/2) = \log(z) - \log(2)$, we deduce that there exists a constant depending only on d and θ such that

$$
\leq C\, |\lambda|^2\, |\log(|\lambda|\, |x|^2)|.
$$

Note that $|z| \leq \mathrm{e}^{-1}$ was used in order to derive $|\log(|z|)| \geq 1$. $\square$

Now, we can derive the compactness of $\mathcal{S}_\lambda - \mathcal{S}_0$.

Lemma 4.3.5. *The operator*

$$
\mathcal{S}_\lambda - \mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)
$$

is compact. Moreover, there exists a constant $C > 0$ depending only on d, θ, and the Lipschitz character of Ω such that for all $\lambda \in \mathrm{S}_\theta$ with $|\lambda| \leq \mathrm{e}^{-2}(\mathrm{diam}(\Omega)^2)^{-1}$ and all $f \in \mathrm{L}^2_\nu(\partial\Omega)$

$$
\|\mathcal{S}_\lambda f - \mathcal{S}_0 f\|_{\mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)} \leq C\, |\lambda|^{\frac{1}{2}}\, \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}.
$$

Proof. Define $\mathcal{S} := \mathcal{S}_\lambda - \mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ and assume for a moment that $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and $\nabla_{\tan}\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^{d^2})$ are compact. Then, if $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2_\nu(\partial\Omega)$ is bounded, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(\mathcal{S} f_{n_k})_{k\in\mathbb{N}}$ converges in $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. Moreover, by compactness of $\nabla_{\tan}\mathcal{S}$ there exists another subsequence $(f_{n_{k_l}})_{l\in\mathbb{N}}$ such that $(\nabla_{\tan}\mathcal{S} f_{n_{k_l}})_{l\in\mathbb{N}}$ converges in $\mathrm{L}^2(\partial\Omega; \mathbb{C}^{d^2})$, which proves the compactness of $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ and hence of $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$. Summarizing, it remains to prove that $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and $\nabla_{\tan}\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^{d^2})$ are compact.

Let us first concentrate on the operator $\mathcal{S}$ as an operator from $\mathrm{L}^2_\nu(\partial\Omega)$ into $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. We will prove that it can be approximated by compact operators in the operator norm. For this purpose, define for $\varepsilon > 0$ the operator

$$\begin{aligned} &\mathcal{S}^\varepsilon : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \\ &\quad [\mathcal{S}^\varepsilon f](p) := \int_{\partial\Omega \setminus B(p,\varepsilon)} \Big\{\Gamma(p-y;\lambda) - \Gamma(p-y;0)\Big\} f(y) \, \mathrm{d}\sigma(y) \end{aligned}$$

and estimate by means of Proposition 1.1.4

$$\begin{aligned} &\|\mathcal{S}f - \mathcal{S}^\varepsilon f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \\ &\leq \sup_{p\in\partial\Omega} \Big\| \Big[\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\Big] \chi_{B(p,\varepsilon)}(p-\cdot) \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \end{aligned}$$

To show that

$$\sup_{p\in\partial\Omega} \Big\| \Big[\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\Big] \chi_{B(p,\varepsilon)}(p-\cdot) \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

let ε be small enough. By Lemma 4.3.4 there exists a constant $C > 0$ depending only on d and θ, such that

$$|\Gamma(x;\lambda) - \Gamma(x;0)| \leq C \begin{cases} |\lambda|^{1/2}, & \text{if } d = 3 \\ |\lambda| \, |x|^{4-d} \, |\log(|\lambda| \, |x|^2)|, & \text{if } d = 4, 6 \\ |\lambda| \, |x|^{4-d}, & \text{if } d = 5, d \geq 7. \end{cases}$$

if $|\lambda| \, |x|^2 \leq \mathrm{e}^{-2}$. Note that the smallness condition on $|\lambda| \, |x|^2$ implies that

$$|\lambda|^{1/2} \, |x| \, |\log(|\lambda| \, |x|^2)| \leq \frac{2}{\mathrm{e}}, \tag{4.29}$$

so that in any dimension

$$|\Gamma(x;\lambda) - \Gamma(x;0)| \leq C \, |\lambda|^{1/2} \, |x|^{3-d}. \tag{4.30}$$

Use this estimate to conclude that

$$\int_{\partial\Omega\cap B(p,\varepsilon)} |\Gamma(p-y;\lambda) - \Gamma(p-y;0)| \,\mathrm{d}\sigma(y) \\ \leq C\,|\lambda|^{1/2} \int_{\partial\Omega\cap B(p,\varepsilon)} \frac{1}{|p-y|^{d-3}} \,\mathrm{d}\sigma(y)$$

and then infer by Lemma 4.3.2 that

$$\leq C\,|\lambda|^{1/2}\,\varepsilon^2.$$

Note that the kernel of $\mathcal{S} - \mathcal{S}^\varepsilon$ lies in $\mathrm{L}^\infty(\partial\Omega;\mathbb{C}^{d\times d}) \times \mathrm{L}^\infty(\partial\Omega;\mathbb{C}^{d\times d})$. Thus, by virtue of Weidmann [98, Thm. 6.11], it is a Hilbert-Schmidt operator and hence compact. We conclude that $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is compact.

We turn to investigate the operator $\nabla_{\tan}\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})$. By virtue of Lemma 1.3.21, we can compute the non-tangential gradient of $\mathcal{S}f$ if a certain non-tangential behavior is known. This indeed holds true, as firstly, by Proposition 4.3.1, the non-tangential limits of $\mathcal{S}_\lambda f$ to $\partial\Omega$ exists σ-a.e. The same is valid for $\mathcal{S}_0 f$ due to the statement below (0.13) in Fabes, Kenig, and Verchota [28]. Secondly, the non-tangential maximal functions of $\mathcal{S}_0 f$, $\nabla\mathcal{S}_0 f$, $\mathcal{S}_\lambda f$, and $\nabla\mathcal{S}_\lambda f$ are in $\mathrm{L}^2(\partial\Omega)$ due to [28, Eq. (0.7)] and [89, Lem. 3.2]. Thirdly, the non-tangential limits of $\nabla\mathcal{S}_0 f$ and $\nabla\mathcal{S}_\lambda f$ to $\partial\Omega$ exist σ-a.e., see [28, Eq. (0.9)] and [89, Lem. 3.3]. Moreover, these results provide the following representation of the non-tangential limits

$$[\partial_i\mathcal{S}_\mu f]_+(p) = \frac{1}{2}\Big\{\nu_i(p)f(p) - \nu_i(p)\langle\nu(p), f(p)\rangle\nu(p)\Big\} \\ + \mathrm{p.v.}\int_{\partial\Omega} \partial_{p_i}[\Gamma(p-y;\mu)]f(y)\,\mathrm{d}\sigma(y) \qquad (\sigma\text{-a.e. } p\in\partial\Omega),$$

where μ is either 0 or λ. Thus, Lemma 1.3.21 is applicable and we find the following identity for $\nabla_{\tan}[\mathcal{S}f]_j$:

$$\nabla_{\tan}[\mathcal{S}f]_j(p) = \sum_{k=1}^d \mathrm{p.v.}\int_{\partial\Omega} \nabla_p[\Gamma_{jk}(p-y;\lambda) - \Gamma_{jk}(p-y;0)]f_k(y)\,\mathrm{d}\sigma(y) \\ - \sum_{k=1}^d \mathrm{p.v.}\int_{\partial\Omega} \langle\nu(p), \nabla_p[\Gamma_{jk}(p-y;\lambda) - \Gamma_{jk}(p-y;0)]f_k(y)\rangle\nu(p)\,\mathrm{d}\sigma(y).$$

By [89, Thm. 2.5] there exists a constant $C > 0$ depending only on d and θ such that for all $x \in \mathbb{R}^d \setminus \{0\}$ with $|\lambda| \, |x|^2 \leq 1/2$

$$|\nabla[\Gamma(x;\lambda) - \Gamma(x;0)]| \leq C \begin{cases} |\lambda| \, |x|^{3-d}, & \text{if } d \geq 7 \text{ or } d = 5, \\ |\lambda| \, |x|^{3-d} \, |\log(|\lambda| \, |x|^2)|, & \text{if } d = 4 \text{ or } 6, \\ |\lambda|^{1/2} \, |x|^{-1}, & \text{if } d = 3. \end{cases}$$

Use (4.29) to conclude that in any dimension

$$|\nabla[\Gamma(x;\lambda) - \Gamma(x;0)]| \leq C \, |\lambda|^{1/2} \, |x|^{2-d}, \tag{4.31}$$

whenever $|\lambda||x|^2 \leq \mathrm{e}^{-2}$. To prove compactness of $\nabla_{\tan}\mathcal{S}$ one proceeds as for $\mathcal{S}$ above and approximates $\nabla_{\tan}\mathcal{S}$ in the operator norm by operators with a truncated kernel. Due to $|\nu| = 1$ it suffices to estimate for sufficiently small $\varepsilon > 0$

$$\Big\| \int_{\partial\Omega \cap B(\cdot,\varepsilon)} |\nabla_\cdot[\Gamma(\cdot - y; 0) - \Gamma(\cdot - y; \lambda)]| \, |f(y)| \, \mathrm{d}\sigma(y) \Big\|_{\mathrm{L}^2(\partial\Omega)}.$$

Appealing to Proposition 1.1.4 and the kernel estimate above, this is controlled by

$$\leq C \, |\lambda|^{1/2} \sup_{p \in \partial\Omega} \int_{\partial\Omega \cap B(p,\varepsilon)} \frac{1}{|p - y|^{d-2}} \, \mathrm{d}\sigma(y) \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}.$$

Use Lemma 4.3.2 to obtain the bound

$$\leq C \, |\lambda|^{1/2} \, \varepsilon \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}.$$

It follows that the right-hand side converges to zero as $\varepsilon \to 0$ and that $\nabla_{\tan}\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})$ is compact.

It remains to give a bound on the operator norm of $\mathcal{S} : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)$ if $|\lambda| \leq \mathrm{e}^{-2}(\operatorname{diam}(\Omega)^2)^{-1}$. As above, we derive

$$\|\mathcal{S}f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq \sup_{p \in \partial\Omega} \|\Gamma(p - \cdot;\lambda) - \Gamma(p - \cdot;0)\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}$$

and

$$\begin{aligned} &\|\nabla_{\tan}\mathcal{S}f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} \\ &\qquad \leq 2 \sup_{p \in \partial\Omega} \|\nabla_p[\Gamma(p - \cdot;\lambda) - \Gamma(p - \cdot;0)]\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \end{aligned}$$

Note that $|\lambda| \leq \mathrm{e}^{-2} \operatorname{diam}(\Omega)^{-2}$ implies that for each $y \in \partial\Omega$

$$|\lambda| \, |p - y|^2 \leq |\lambda| \operatorname{diam}(\Omega)^2 \leq \mathrm{e}^{-2}.$$

Thus, in both estimates, we can appeal to (4.30) and (4.31) as well as to Lemma 4.3.2 to conclude that there exists a constant $C > 0$, depending only on d, θ, r_0, and the Lipschitz character of Ω, such that

$$\|\mathcal{S} f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C \, |\lambda|^{1/2} \, \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}$$

and

$$\|\nabla_{\tan} \mathcal{S} f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} \leq C \, |\lambda|^{1/2} \, \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}.$$

This concludes the proof. □

Now, we are in the position to prove Theorem 4.1.9.

Proof of Theorem 4.1.9. By the paragraph preceding Theorem 4.15 in [28], the operator $\mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ is invertible and thus a Fredholm operator. By Lemma 4.3.5, the operator $\mathcal{S}_\lambda - \mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ is compact, so that $\mathcal{S}_\lambda = \mathcal{S}_\lambda - \mathcal{S}_0 + \mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ is a Fredholm operator as well, see Schechter [85, Thm. 5.10]. By the very same theorem, the Fredholm index of $\mathcal{S}_\lambda$ is zero because the Fredholm index of $\mathcal{S}_0$ is zero. Moreover, as $\mathcal{S}_\lambda$ is injective by Proposition 4.3.1 it is also surjective. Consequently, if $g \in \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ there exists a unique $f \in \mathrm{L}^2_\nu(\partial\Omega)$ such that

$$\mathcal{S}_\lambda f = g.$$

By uniqueness of the L^2-Dirichlet problem for the Stokes resolvent, $\mathcal{S}_\lambda f$ coincides with the solution given by Theorem 4.1.7. By the invertibility of $\mathcal{S}_0 : \mathrm{L}^2_\nu(\partial\Omega) \to \mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega; \mathbb{C}^d)$ and the triangle inequality

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq \|\mathcal{S}_0^{-1}\| \Big\{ \|\mathcal{S}_\lambda f - \mathcal{S}_0 f\|_{\mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)} + \|\mathcal{S}_\lambda f\|_{\mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)} \Big\}.$$

Here, we used the notation $\|\mathcal{S}_0^{-1}\|$ for $\|\mathcal{S}_0^{-1}\|_{\mathcal{L}(\mathrm{L}^2_\nu(\partial\Omega) \cap \mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d), \mathrm{L}^2_\nu(\partial\Omega))}$. If λ is small, we can absorb the first summand on the right-hand side into the left-hand side by means of Lemma 4.3.5, to derive

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C \|\mathcal{S}_\lambda f\|_{\mathrm{W}^{1,2}(\partial\Omega;\mathbb{C}^d)}.$$

If $|\lambda|$ is large, use the estimate in Proposition 4.3.3 to get

$$\|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \leq C\Big\{\|\nabla_{\tan}\mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^{d^2})} + |\lambda|^{\frac{1}{2}}\,\|\mathcal{S}_\lambda f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} + |\lambda|\,\|\langle\nu,\mathcal{S}_\lambda f\rangle\|_{\mathrm{W}^{1,2}(\partial\Omega)^*}\Big\}. \qquad \square$$

CHAPTER 5

The Stokes operator

The purpose of this chapter is twofold. On the one hand, it serves as a collection of properties of the Stokes operator on bounded Lipschitz domains that are well-known to experts and further properties that are “easy to see”. We will provide proofs of these facts.

On the other hand, we establish new results, that open up the door for an L^p-theory of the Navier-Stokes equations on three dimensional bounded Lipschitz domains. The fact that the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ has maximal L^q-regularity and various types of L^p-L^q-estimates of the Stokes semigroup are the main results of this chapter, see Theorems 5.2.24 and 5.2.22.

Unfortunately, we did not succeed to prove gradient estimates for the Stokes semigroup for $p > 2$. Nevertheless, several attempts to achieve these, have lead to further insights about possible approaches to these estimates. So, in Section 5.3, we present two possible approaches to establish gradient estimates. The first of these approaches leads to the result that the boundedness of the H^∞-calculus of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ implies the validity of gradient estimates. In the second approach, we first present a proof of the gradient estimates in the elliptic situation and then we point out the differences to the situation of the Stokes equations.

Many properties rely on the Helmholtz projection and a representation formula of this projection by means of the weak Neumann Laplacian. Although the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for

$3/2 - \varepsilon < p < 3 + \varepsilon$ and $d \geq 3$ is already known due to Fabes, Mendez, and Mitrea [29], we present a different proof. With our proof, we obtain a smaller interval for the boundedness of the Helmholtz projection, namely,

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon.$$

However, our proof also covers the case $d = 2$, thereby revealing the astonishing fact, that the interval of p's, where the Helmholtz projection is bounded, is independent of d, except for the case $d = 2$.

Furthermore, our approach enables us to prove the boundedness of the Helmholtz projection on certain unbounded Lipschitz domains, i.e., domains above the graph of a Lipschitz function. In view of the generalization of Shen's L^p-extrapolation theorem to unbounded domains, that are not the whole space, see Theorem 3.1.2, it serves as a first application of this result.

5.1 The Helmholtz projection on Lipschitz domains

As was already introduced in Subsection 1.1.4, the Helmholtz projection on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ is the orthogonal projection $\mathbb{P}$ onto $\mathrm{L}^2_\sigma(\Omega)$. As $\mathrm{L}^2_\sigma(\Omega)$ is a closed subspace of the Hilbert space $\mathrm{L}^2(\Omega; \mathbb{C}^d)$, the Helmholtz projection exists for every open set Ω and is bounded with operator norm being one. The extension of this operator to $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ is a non-trivial task, see for example Theorem 1.1.19. If Ω is unbounded, there exist even smooth domains, which are "sector-like" in a certain sense, such that the Helmholtz projection does not extend to a bounded operator from $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ onto $\mathrm{L}^p_\sigma(\Omega)$ for certain p's. More precisely, if $\Omega \subset \mathbb{R}^2$ is has a smooth boundary and satisfies $\Omega \setminus B(0, R) = \mathrm{S}_\theta \setminus B(0, R)$ for some $R > 0$ and $\theta \in (0, \pi)$, then the boundedness of the Helmholtz projection fails on $\mathrm{L}^p(\Omega; \mathbb{R}^2)$ whenever

$$p \notin [q_\theta, q_\theta'], \quad q_\theta := \frac{2}{1 + \pi/(2\theta)}, \quad \frac{1}{q_\theta} + \frac{1}{q_\theta'} = 1.$$

This was proven by Maslennikova and Bogovskiĭ in [68]. Note that the supremum of the q_θ's is given by

$$\sup_{\theta \in (0,\pi)} q_\theta = \frac{2}{1 + \pi/(2\pi)} = \frac{4}{3}$$

and that this coincides with $2d/(d+1)$ if $d = 2$. Since such domains will be covered by the theory presented in this section, the example of Maslennikova and Bogovskiĭ shows the sharpness of our results for two dimensional special Lipschitz domains. We start with the definition of homogeneous Sobolev spaces.

Definition 5.1.1. For $1 < p < \infty$, define the *homogeneous Sobolev space* $\dot{\mathrm{W}}^{1,p}(\Omega)$ as the space of equivalence classes of functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, such that $\nabla u \in \mathrm{L}^p(\Omega)$ and where two functions are identified if they coincide modulo an additional constant. The space is endowed with the norm

$$\|u\|_{\dot{\mathrm{W}}^{1,p}(\Omega)} := \|\nabla u\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}.$$

Remark 5.1.2. For a thorough investigation of homogeneous Sobolev spaces, we refer to Galdi [34, Ch. II]. For example, [34, Lem. II.6.2] states that $\dot{\mathrm{W}}^{1,2}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_\Omega \langle \nabla u, \nabla v \rangle \, \mathrm{d}x.$$

Next, we define the divergence of a function $f \in \mathrm{L}^p(\Omega; \mathbb{C}^d)$ as an antilinear functional, acting on $\dot{\mathrm{W}}^{1,p'}(\Omega)$, with p' being the Hölder conjugate exponent of p. This is done by defining

$$[\operatorname{div}(f)](v) := - \int_\Omega \langle f, \nabla v \rangle \, \mathrm{d}x,$$

which is an element of the dual space of antilinear functionals on $\dot{\mathrm{W}}^{1,p'}(\Omega)$, for short, of $(\dot{\mathrm{W}}^{1,p'}(\Omega))^*$.

Additionally, define the weak Neumann Laplacian Δ_N as the operator

$$\Delta_N : \dot{\mathrm{W}}^{1,2}(\Omega) \to (\dot{\mathrm{W}}^{1,2}(\Omega))^*, \quad u \mapsto \Big[\dot{\mathrm{W}}^{1,2}(\Omega) \ni v \mapsto - \int_\Omega \langle \nabla u, \nabla v \rangle \, \mathrm{d}x\Big].$$

By the lemma of Lax–Milgram, we derive that for any antilinear functional $g \in (\dot{\mathrm{W}}^{1,2}(\Omega))^*$ there exists a unique function $u \in \dot{\mathrm{W}}^{1,2}(\Omega)$ such that

$$-\Delta_N u = g.$$

Because $\operatorname{div}(f)$ is an element of $(\dot{\mathrm{W}}^{1,2}(\Omega))^*$ for every $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$, we deduce that $(-\Delta_N)^{-1} \operatorname{div}(f)$ is in $\dot{\mathrm{W}}^{1,2}(\Omega)$, so that the expression $\nabla(-\Delta_N)^{-1} \operatorname{div}(f)$ defines an element of $\mathrm{L}^2(\Omega; \mathbb{C}^d)$.

As the following lemma shows, this expression is crucial for obtaining a representation formula of $\mathbb{P}$.

Lemma 5.1.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open set. Then for all $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ the following identity holds*

$$\mathbb{P} f = f + \nabla(-\Delta_N)^{-1} \operatorname{div}(f).$$

Proof. We begin the proof by defining the operator

$$\widetilde{\mathbb{P}} : \mathrm{L}^2(\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\Omega; \mathbb{C}^d), \quad f \mapsto f + \nabla(-\Delta_N)^{-1} \operatorname{div}(f).$$

The aim is to show that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ coincide.

Claim 1: $\widetilde{\mathbb{P}}$ is bounded.

Denote the action of $-\Delta_N u$ under $v \in \dot{\mathrm{W}}^{1,2}(\Omega)$ by $[-\Delta_N u](v)$. By definition of the weak Neumann Laplacian and the divergence, we calculate for $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$

$$\begin{aligned}\int_\Omega |\nabla(-\Delta_N)^{-1} \operatorname{div}(f)|^2 \, \mathrm{d}x &= \Big[-\Delta_N(-\Delta_N)^{-1} \operatorname{div}(f)\Big]((-\Delta_N)^{-1} \operatorname{div}(f)) \\ &= -\int_\Omega \langle f, \nabla(-\Delta_N)^{-1} \operatorname{div}(f)\rangle \, \mathrm{d}x.\end{aligned}$$

Hölder's inequality finally yields

$$\leq \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \|\nabla(-\Delta_N)^{-1} \operatorname{div}(f)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}.$$

Claim 2: $\operatorname{div}(\widetilde{\mathbb{P}} f) = 0$ **for all** $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$**.**

Let $v \in \dot{\mathrm{W}}^{1,2}(\Omega)$. By means of the definition of $\operatorname{div}(f)$, we have

$$[\operatorname{div}(\widetilde{\mathbb{P}} f)](v) = -\int_\Omega \langle \widetilde{\mathbb{P}} f, \nabla v\rangle \, \mathrm{d}x.$$

Employing the definition of $\widetilde{\mathbb{P}}f$ on the right-hand side delivers

$$= -\int_\Omega \langle f, \nabla v\rangle \, \mathrm{d}x - \int_\Omega \langle \nabla(-\Delta_N)^{-1} \operatorname{div}(f), \nabla v\rangle \, \mathrm{d}x.$$

The second integral on the right-hand side coincides with the expression $[-\Delta_N(-\Delta_N)^{-1} \operatorname{div}(f)](v) = [\operatorname{div}(f)](v)$, so that by definition of $\operatorname{div}(f)$

$$= -\int_\Omega \langle f, \nabla v\rangle \, \mathrm{d}x + \int_\Omega \langle f, \nabla v\rangle \, \mathrm{d}x = 0.$$

Claim 3: $\widetilde{\mathbb{P}}$ is an orthogonal projection.
From Claim 2, we derive for $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$

$$\widetilde{\mathbb{P}}\widetilde{\mathbb{P}}f = \widetilde{\mathbb{P}}f + \nabla(-\Delta_N)^{-1} \operatorname{div}(\widetilde{\mathbb{P}}f) = \widetilde{\mathbb{P}}f,$$

so that $\widetilde{\mathbb{P}}$ is a projection. Let additionally $g \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$. Then by definition of $\widetilde{\mathbb{P}}$

$$\begin{aligned}\int_\Omega \langle \widetilde{\mathbb{P}}f, [\widetilde{\mathbb{P}} - \mathrm{Id}]g\rangle \, \mathrm{d}x = \int_\Omega &\langle f, \nabla(-\Delta_N)^{-1} \operatorname{div}(g)\rangle \, \mathrm{d}x \\ &+ \int_\Omega \langle \nabla(-\Delta_N)^{-1} \operatorname{div}(f), \nabla(-\Delta_N)^{-1} \operatorname{div}(g)\rangle \, \mathrm{d}x.\end{aligned}$$

Write the second integral on the right-hand side by means of the weak Neumann Laplacian to derive

$$\begin{aligned}= \int_\Omega &\langle f, \nabla(-\Delta_N)^{-1} \operatorname{div}(g)\rangle \, \mathrm{d}x \\ &+ \Big[-\Delta_N(-\Delta_N)^{-1} \operatorname{div}(f)\Big]((-\Delta_N)^{-1} \operatorname{div}(g)).\end{aligned}$$

By definition of $\operatorname{div}(f)$, this coincides with

$$\begin{aligned}= \int_\Omega &\langle f, \nabla(-\Delta_N)^{-1} \operatorname{div}(g)\rangle \, \mathrm{d}x \\ &- \int_\Omega \langle f, \nabla(-\Delta_N)^{-1} \operatorname{div}(g)\rangle \, \mathrm{d}x \\ = 0&.\end{aligned}$$

Claim 4: The operators $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ coincide.

It is classical, that on arbitrary open sets the orthogonal complement of $\mathrm{L}^2_\sigma(\Omega)$ is given by

$$G(\Omega) := \{\nabla u : u \in \dot{\mathrm{W}}^{1,2}(\Omega)\},$$

see Sohr [90, Lem. II.2.5.1]. In Claim 2, we have seen that the range $\mathcal{R}(\widetilde{\mathbb{P}})$ of $\widetilde{\mathbb{P}}$ is contained in $G(\Omega)^\perp$, so that $\mathcal{R}(\widetilde{\mathbb{P}}) \subset \mathrm{L}^2_\sigma(\Omega)$. On the other hand, $\mathcal{R}(\widetilde{\mathbb{P}} - \mathrm{Id}) = \mathcal{R}(\nabla(-\Delta_N)^{-1} \operatorname{div}) \subset G(\Omega)$. By Claim 3, we have $\mathcal{R}(\widetilde{\mathbb{P}})^\perp = \mathcal{R}(\widetilde{\mathbb{P}} - \mathrm{Id})$, so that $\mathrm{L}^2_\sigma(\Omega) \subset \mathcal{R}(\widetilde{\mathbb{P}})$. We conclude that $\widetilde{\mathbb{P}}$ is an orthogonal projection with $\mathcal{R}(\widetilde{\mathbb{P}}) = \mathrm{L}^2_\sigma(\Omega)$. Because this holds certainly true for the Helmholtz projection, both projections $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ must coincide. □

Remark 5.1.4. I learned of the validity of the previous lemma due to a private conversation with M. Geissert.

In the following, we are concerned with establishing the L^p-boundedness of $\mathbb{P}$ if $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is either a bounded Lipschitz domain or a special Lipschitz domain. The latter is defined as follows.

Definition 5.1.5. An open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is called a *special Lipschitz domain* if there exists a Lipschitz continuous function $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega = \{(x', x_d) \in \mathbb{R}^d : x_d > \eta(x')\}.$$

By Lemma 5.1.3 the restriction of the Helmholtz projection onto the space $\mathrm{L}^2(\Omega; \mathbb{C}^d) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$ extends to a bounded operator on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ if and only if this is the case for the operator $\nabla(-\Delta_N)^{-1} \operatorname{div}$. Thus, it is the aim to apply Theorem 3.1.2 to this operator with $X = Y = \mathbb{C}^d$.

To prove the validity of the weak reverse Hölder estimates, recall the non-tangential maximal functions N_a, $a > 0$, defined in (4.13). The following lemma will be crucial to prove these weak reverse Hölder estimates. If $d \geq 3$, we refer to Shen [89, p. 418] for a proof, see also Wei and Zhang [97, Lem. 3.3]. The two dimensional case was treated by Barton [7, Lem. 3.3].

Lemma 5.1.6. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. If $\varphi : \Omega \to \mathbb{C}^N$ is measurable, then there exists a constant $C > 0$ depending only on d and the Lipschitz character of Ω such that*

$$\|\varphi\|_{\mathrm{L}^{2d/(d-1)}(\Omega; \mathbb{C}^N)} \leq C \|N_1 \varphi\|_{\mathrm{L}^2(\partial\Omega)}.$$

In the interior of Ω, weak reverse Hölder estimates are a consequence of de Giorgi's theorem, see Giaquinta and Martinazzi [37, Thm. 8.13], applied to $\partial_i(-\Delta_N)^{-1}\operatorname{div}(f)$. Despite that, we will present a different approach for two reasons. The first is, that it serves as a blue print for a technique used in several other proofs in this chapter. The second is, that the interior case for the Laplacian is the easiest situation, that can be considered. Thus, one gets the ideas of the major steps in this technique and a feeling for where the subtleties are in more complicated situations. Note that this technique was used by Shen in order to derive weak reverse Hölder estimates for the Stokes resolvent, see [89, Lem. 6.1], and by Wei and Zhang [97, Prop. 3.1] for the resolvent estimates of constant coefficient elliptic systems subject to Neumann boundary conditions.

Proposition 5.1.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded or special Lipschitz domain and let $r > 0$ and $x_0 \in \Omega$ be such that $B(x_0, 3r) \subset \Omega$. Then there exists a constant $C > 0$ depending only on d such that for all $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ with $f = 0$ on $B(x_0, 3r)$, we have for $u := (-\Delta_N)^{-1}\operatorname{div}(f)$*

$$\Big(\frac{1}{r^d}\int_{B(x_0,r)} |\nabla u|^{\frac{2d}{d-1}} \,\mathrm{d}x\Big)^{\frac{d-1}{2d}} \leq C\Big(\frac{1}{r^d}\int_{B(x_0,2r)} |\nabla u|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}$$

holds.

Proof. First of all, note that for all $v \in \mathrm{C}_c^\infty(B(x_0, 3r))$

$$0 = -\int_{B(x_0,3r)} \langle f, \nabla v\rangle \,\mathrm{d}x = \int_{B(x_0,3r)} \langle \nabla u, \nabla v\rangle \,\mathrm{d}x,$$

so that by means of Weyl's lemma, see Giaquinta and Martinazzi [37, Lem. 1.16], u is harmonic in $B(x_0, 3r)$ and thus smooth. Next, consider the rescaled function $u_r(x) := u(rx)$ on $B(r^{-1}x_0, 3)$. Choose $s \in [1, 2]$ and note that

$$\Big(\int_{B(r^{-1}x_0,1)} |\nabla u_r|^{\frac{2d}{d-1}} \,\mathrm{d}x\Big)^{\frac{d-1}{d}} \leq \Big(\int_{B(r^{-1}x_0,s)} |\nabla u_r|^{\frac{2d}{d-1}} \,\mathrm{d}x\Big)^{\frac{d-1}{d}}.$$

Then, appeal to Lemma 5.1.6 to deduce

$$\leq C\int_{\partial B(r^{-1}x_0,s)} |N_1 \nabla u_r|^2 \,\mathrm{d}\sigma_s,$$

where σ_s denotes the surface measure on $\partial B(r^{-1}x_0, s)$. By smoothness of u_r on $B(r^{-1}x_0, 3)$, we find $(\nabla u_r)^* \in \mathrm{L}^2(\partial B(r^{-1}x_0, s))$. Hence, for each $s \in [1, 2]$, u_r solves the L^2-Neumann problem with $g := \langle \nu_s, \nabla u \rangle$, where ν_s is the outward unit normal to $\partial B(r^{-1}x_0, s)$. By means of Theorem 4.1.3 together with Lemma 4.1.15, we obtain the estimate

$$\leq C \int_{\partial B(r^{-1}x_0,s)} |\nabla u_r|^2 \, \mathrm{d}\sigma_s,$$

where C depends only on d, r_0 (which belongs to $B(r^{-1}x_0, s)$ as a Lipschitz domain via Definition 1.3.1), and the Lipschitz character of $B(r^{-1}x_0, s)$. Moreover, since $s \in [1, 2]$ is away from zero, the constant C can be chosen to be uniform in s (note that the Lipschitz character of a domain is scaling invariant, so that we have to ensure that r_0 is not going to be arbitrarily small. This is ensured if s is away from zero). Next, integrate the whole inequality over $s \in [1, 2]$ with respect to the Lebesgue measure on $\mathbb{R}$. The left-hand side being constant in s, we derive

$$\left(\int_{B(r^{-1}x_0,1)} |\nabla u_r|^{\frac{2d}{d-1}} \, \mathrm{d}x \right)^{\frac{d-1}{d}} \leq C \int_1^2 \int_{\partial B(r^{-1}x_0,s)} |\nabla u_r|^2 \, \mathrm{d}\sigma_s \, \mathrm{d}s.$$

Next, compare σ_s and the $(d-1)$-dimensional Hausdorff measure m_{d-1} to get

$$\leq C C_d \int_1^2 \int_{\partial B(r^{-1}x_0,s)} |\nabla u_r|^2 \, \mathrm{d}m_{d-1} \, \mathrm{d}s,$$

where C_d is the comparison constant of σ_s and m_{d-1}, which depends only on d. By means of the co-area formula, Theorem 1.3.22, with $\mathfrak{g} := |\nabla u_r|^2 \chi_{B(r^{-1}x_0,2)}$ and the Lipschitz continuous function

$$\mathbb{R}^d \ni x \mapsto |x - r^{-1}x_0|,$$

we can estimate the integral on the right-hand side by

$$d^{1/2} \int_{B(r^{-1}x_0,2)} |\nabla u_r|^2 \, \mathrm{d}x.$$

Taking the square root of the resulting inequality, we altogether obtain

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{B(x_0,r)} |\nabla u|^{\frac{2d}{d-1}} \, \mathrm{d}x\Big)^{\frac{d-1}{2d}} &= \frac{1}{r}\Big(\int_{B(r^{-1}x_0,1)} |\nabla u_r|^{\frac{2d}{d-1}} \, \mathrm{d}x\Big)^{\frac{d-1}{2d}} \\ &\leq \frac{[CC_d]^{1/2}d^{1/4}}{r}\Big(\int_{B(r^{-1}x_0,2)} |\nabla u_r|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} \\ &= [CC_d]^{1/2}d^{1/4}\Big(\frac{1}{r^d}\int_{B(x_0,2r)} |\nabla u|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}. \ \square\end{aligned}$$

Recall the sets $U_{x_0,r} = \{x_0\} + R_{x_0}^{-1}D(r)$ defined before (1.5) and recall that $\Omega \cap U_{x_0,r} = \Omega \cap [\{x_0\} + R_{x_0}^{-1}D_{\eta_{x_0}}(r)]$, see Definition 1.3.1. To establish the weak reverse Hölder estimates on balls centered on $\partial\Omega$, we will make a detour by establishing the estimates on $\Omega \cap U_{x_0,r}$ first. Here, the goal is to imitate the proof of Proposition 5.1.7. However, there is one critical point, we have to discuss for this imitation. Recall that we used that u_r solves the L^2-Neumann problem on $B(r^{-1}x_0, s)$ with $\langle \nu_s, \nabla u_r\rangle$ being the boundary data. Here, we were not in any trouble, because of the smoothness of u_r on all of $B(r^{-1}x_0, 3)$.

Letting now $u := (-\Delta_N)^{-1} \operatorname{div}(f)$, where $f = 0$ on $\Omega \cap U_{x_0,3r}$, we see that u in general is not smooth in a neighborhood of $\Omega \cap U_{x_0,sr}$. Additionally, since ∇u is merely in $\mathrm{L}^2(\Omega \cap U_{x_0,3r})$ one cannot even take its trace at the boundary of $\Omega \cap U_{x_0,sr}$ in order to define the boundary data required for the L^2-Neumann problem. Instead, we do the following. Since

$$\int_{\Omega\cap U_{x_0,3r}} \langle \nabla u, \nabla \psi\rangle \, \mathrm{d}x = 0 \qquad (\psi \in \mathrm{C}_c^\infty(\Omega \cap U_{x_0,3r})),$$

an application of Weyl's lemma, see [37, Lem. 1.16], shows that u has a representative $\mathbf{u}$ that is harmonic and thus smooth in $\Omega \cap U_{x_0,3r}$. We see, that we have no trouble with evaluating $\mathbf{u}$ on the boundary of $\Omega \cap U_{x_0,sr}$ that lies in the interior of Ω, i.e., on $\Omega \cap \partial U_{x_0,sr}$. Moreover, since u lies in the domain of the Neumann Laplacian, its Neumann data on $\partial\Omega$ should vanish (in a certain sense). Thus, it seems reasonable, that if $\mathbf{u}$ solves an L^2-Neumann problem on $\Omega \cap U_{x_0,sr}$ at all, then, this is the L^2-Neumann problem corresponding to the data

$$g := \begin{cases} \langle \nu_{U_{x_0,sr}}, \nabla \mathbf{u}\rangle, & \text{on } \Omega \cap \partial U_{x_0,sr} \\ 0, & \text{on } \partial\Omega \cap U_{x_0,sr}, \end{cases} \tag{5.1}$$

where $\nu_{U_{x_0,sr}}$ denotes the outward unit normal to $\partial U_{x_0,sr}$. Note that it is not at all clear, whether $g \in \mathrm{L}^2(\partial[\Omega \cap U_{x_0,sr}])$. However, by an integration argument, we will see that this holds true for almost every $s \in [1,2]$.

The following lemma shows that $\mathbf{u}$ solves the L^2-Neumann problem with boundary data defined in (5.1), whenever this lies in $\mathrm{L}^2(\partial[\Omega \cap U_{x_0,sr}])$. To formulate this lemma, recall that L^2_0 means L^2-integrable with mean zero and that r_0 and M denote the quantities corresponding to Ω as a bounded Lipschitz domain, see Definition 1.3.1.

Lemma 5.1.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, and let $0 < r \leq r_0/3$ and $x_0 \in \partial\Omega$. Let $f \in \mathrm{L}^2(\Omega;\mathbb{C}^d)$ with $f = 0$ in $\Omega \cap U_{x_0,3r}$ and $u := (-\Delta_N)^{-1} \operatorname{div}(f)$. Then, with the definitions above, $g \in \mathrm{L}^2_0(\partial[\Omega \cap U_{x_0,sr}])$ for almost every $s \in [1,2]$ and $\mathbf{u}$ solves the L^2-Neumann problem in $U_{x_0,sr}$ with boundary data g for these s.*

Proof. Apply the co-area formula, Theorem 1.3.22, with the function $\mathfrak{g} := |\nabla \mathbf{u}|^2 \chi_{\Omega \cap U_{x_0,2r}}$ and the Lipschitz continuous function

$$\mathbb{R}^d \ni x \mapsto s \quad \text{iff} \quad x \in \partial U_{x_0,sr}$$

and let this function be zero at $r^{-1}x_0$. Due to the equivalence of the $(d-1)$-dimensional Hausdorff measure m_{d-1} and the surface measure $\sigma_{U_{x_0,sr}}$ on $\partial U_{x_0,sr}$, there exists a number $C > 0$ depending only on d, M, and r such that

$$\int_0^2 \int_{\Omega \cap \partial U_{x_0,sr}} |\nabla \mathbf{u}|^2 \, \mathrm{d}\sigma_{U_{x_0,sr}} \, \mathrm{d}s \leq C \int_{U_{x_0,2r}} |\nabla \mathbf{u}|^2 \, \mathrm{d}x < \infty.$$

Consequently, there exists a set $\mathcal{N} \subset [1,2]$ of measure zero, such that $g \in \mathrm{L}^2(\partial[\Omega \cap U_{x_0,sr}])$ for all $s \in [1,2] \setminus \mathcal{N}$. Fix s suchlike in the following.

What we would like to do next, is to manipulate

$$\int_{\Omega \cap U_{x_0,sr}} \langle \nabla \mathbf{u}, \nabla \psi \rangle \, \mathrm{d}x$$

for $\psi \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ by means of a partial integration. Unfortunately, we cannot do that directly due to the lack of integrability of the second derivatives of $\mathbf{u}$. The Neumann boundary conditions are realized in the sense of the validity of a suitable integration by parts. For example

$$\int_{\Omega \cap U_{x_0,3r}} \langle \nabla u, \nabla \psi \rangle \, \mathrm{d}x = 0 \tag{5.2}$$

holds true for all $\psi \in \mathrm{W}^{1,2}_{\Omega\cap\partial U_{x_0,3r}}(\Omega \cap U_{x_0,3r})$, which is the Sobolev space adapted to mixed boundary conditions from Definition 1.1.11. These are valid test functions, as one can simply extend them to all of Ω by zero. We will describe in the following, how to use (5.2) to perform the desired integration by parts above.

For this purpose, let $A(s) := \partial\Omega \cap \partial U_{x_0,sr}$ denote the points in the interface between the cylinder barrel of $U_{x_0,sr}$ and $\partial\Omega$. If $d = 2$, $A(s)$ consists simply of two points and, if $d \geq 3$, then $A(s)$ is the bi-Lipschitz image under the coordinate function $\Phi_{x_0,sr}$ (that corresponds to $U_{x_0,sr}$ via (1.5)) of the sphere $\partial B'(0, sr) \subset \mathbb{R}^{d-1}$. Thus, in the case $d \geq 3$, the $(d-2)$-dimensional Hausdorff measure of $A(s)$ can be compared with

$$m_{d-2}(\{x \in \mathbb{R}^d : x_d = 0, |x'| = sr\}) < \infty. \tag{5.3}$$

Let $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d \setminus A(s))$. Since $A(s)$ is closed and since the support of ψ is compact, one finds $0 < \kappa < r$ such that

$$A_\kappa(s) := \{x \in \mathbb{R}^d : \operatorname{dist}(x, A(s)) < \kappa\} \subset \operatorname{supp}(\psi)^c.$$

Let ϑ be a smooth cut-off function, being one in $\operatorname{supp}(\psi) \cap \Omega \cap U_{x_0,sr}$ and zero in $[\Omega \cap U^c_{x_0,sr+\kappa/4}] \cup A_{\kappa/2}(s)$. Since κ is supposed to be less than r and since $s \in [1, 2]$, we find that $\psi\vartheta$ vanishes in a neighborhood of $\Omega \cap \partial U_{x_0,3r}$, i.e., that $\psi\vartheta \in \mathrm{W}^{1,2}_{\Omega\cap\partial U_{x_0,3r}}(\Omega \cap U_{x_0,3r})$. See Figure 5 for a draft.

For $h > 0$ define

$$U^{h,+}_{x_0,tr} := \{x_0\} + R^{-1}_{x_0} D_{\eta_{x_0}+h}(tr) \qquad (t \in [1, 3]).$$

This set is almost the set $\Omega \cap U_{x_0,tr}$ but the portion of the boundary which comes from $\partial\Omega$ is shifted inside Ω by h. Denote the outward unit normal to $\partial U^{h,+}_{x_0,tr}$ by ν^h_t and the surface measure on $\partial U^{h,+}_{x_0,tr}$ by σ^h_t. Appealing to (5.2) and the dominated convergence theorem

$$0 = \lim_{h \searrow 0} \int_{U^{h,+}_{x_0,3r}} \langle \nabla \mathbf{u}, \nabla[\psi\vartheta]\rangle \, \mathrm{d}x.$$

Integration by parts then yields

$$= \lim_{h \searrow 0} \Bigg[- \int_{U^{h,+}_{x_0,3r}} \Delta \mathbf{u} \overline{\psi} \vartheta \, \mathrm{d}x + \int_{\partial U^h_{x_0,3r}} \langle \nu^{h,+}_3, \nabla \mathbf{u} \rangle \overline{\psi} \vartheta \, \mathrm{d}\sigma^h_3 \Bigg].$$

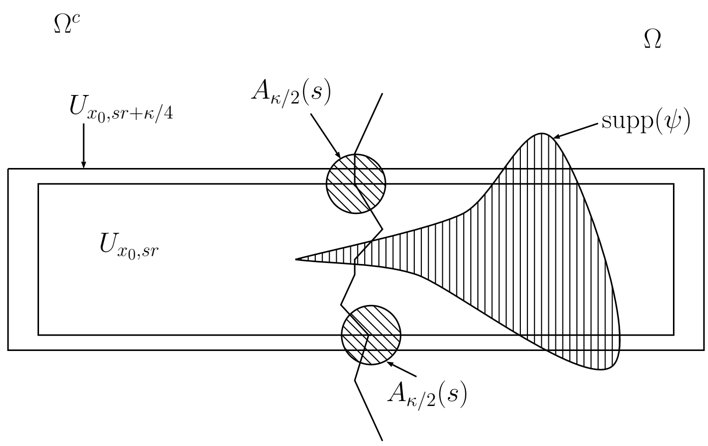

Figure 5: The function ϑ is supposed to be one in the intersection of $\mathrm{supp}(\psi)$ with Ω and the inner cylinder and zero on the union of $A_{\kappa/2}(s)$ and the portion of the complement of the outer cylinder, that lies inside Ω.

The first integral vanishes since $\mathbf{u}$ is harmonic. For the second integral, decompose $\partial U^{h,+}_{x_0,3r}$ into $\partial U^{h,+}_{x_0,3r} \cap U_{x_0,sr}$ and $\partial U^{h,+}_{x_0,3r} \cap U^c_{x_0,sr}$. By construction, $\psi\vartheta$ vanishes on $\partial U^{h,+}_{x_0,3r} \cap U^c_{x_0,sr}$ if h is small enough. Recall that ϑ is one on $\mathrm{supp}(\psi) \cap \Omega \cap U_{x_0,sr}$, so that the limit turns into

$$= \lim_{h \searrow 0} \int_{\partial U^{h,+}_{x_0,3r} \cap U_{x_0,sr}} \langle \nu^h_s, \nabla \mathbf{u} \rangle \overline{\psi} \, \mathrm{d}\sigma^h_3.$$

Finally, note that since ψ vanishes on $A_\kappa(s)$, the integration is actually performed on $[\partial U^{h,+}_{x_0,3r} \cap U_{x_0,sr}] \setminus A_{\kappa(s)}$. Since σ^h_s and σ^h_3 coincide on this set, we have

$$= \lim_{h \searrow 0} \int_{\partial U^{h,+}_{x_0,3r} \cap U_{x_0,sr}} \langle \nu^h_s, \nabla \mathbf{u} \rangle \overline{\psi} \, \mathrm{d}\sigma^h_s.$$

Starting an integration by parts again, but this time on $U_{x_0,sr}$, the dominated convergence theorem yields

$$\begin{aligned}\int_{\Omega\cap U_{x_0,sr}} \langle \nabla \mathbf{u}, \nabla\psi\rangle \,\mathrm{d}x &= \lim_{h\searrow 0}\int_{U^{h,+}_{x_0,sr}} \langle \nabla \mathbf{u}, \nabla\psi\rangle \,\mathrm{d}x \\ &= \lim_{h\searrow 0}\int_{\partial U^{h,+}_{x_0,sr}\setminus U_{x_0,sr}} \langle \nu_s^h, \nabla \mathbf{u}\rangle \overline{\psi}\,\mathrm{d}\sigma_s^h \\ &\quad + \lim_{h\searrow 0}\int_{\partial U^{h,+}_{x_0,sr}\cap U_{x_0,sr}} \langle \nu_s^h, \nabla \mathbf{u}\rangle \overline{\psi}\,\mathrm{d}\sigma_s^h.\end{aligned}$$

The second limit vanishes due to the previous calculation combined with

$$\partial U^{h,+}_{x_0,3r}\cap U_{x_0,sr} = \partial U^{h,+}_{x_0,sr}\cap U_{x_0,sr}.$$

For the first integral, note that ν_s^h coincides with $\nu_{U_{x_0,sr}}$ on $\partial U^{h,+}_{x_0,sr}\setminus U_{x_0,sr}$. Moreover, since $\psi = 0$ on $A_\kappa(s)$, the first integration is acutally performed on $[\partial U^{h,+}_{x_0,sr}\setminus U_{x_0,sr}]\setminus A_\kappa(s)$. Note that σ_s^h and $\sigma_{U_{x_0,sr}}$ coincide on this set and that we have $[\partial U^{h,+}_{x_0,sr}\setminus U_{x_0,sr}]\setminus A_\kappa(s) = \partial[\Omega\cap U_{x_0,sr}]\setminus A_\kappa(s)$ for all h that are small enough. This implies the convergence of the first integral. Thus,

$$\int_{\Omega\cap U_{x_0,sr}} \langle \nabla \mathbf{u}, \nabla\psi\rangle \,\mathrm{d}x = \int_{\Omega\cap\partial U_{x_0,sr}} \langle \nu_{U_{x_0,sr}}, \nabla \mathbf{u}\rangle \overline{\psi}\,\mathrm{d}\sigma_{U_{x_0,sr}}. \tag{5.4}$$

Since s is chosen such that

$$\int_{\Omega\cap\partial U_{x_0,sr}} |\nabla \mathbf{u}|^2 \,\mathrm{d}\sigma_{U_{x_0,sr}}$$

is finite, we conclude by density that (5.4) holds for all $\psi \in \mathrm{W}^{1,2}_0(\mathbb{R}^d\setminus A(s))$. Moreover, by (5.3), the $(d-2)$-dimensional Hausdorff measure of $A(s)$ is finite for $d \geq 3$. Thus, an application of Adams and Hedberg [2, Thm. 5.1.9] shows that $A(s)$ is "thin" in a certain sense (the $(1,2)$-capacity of $A(s)$ is zero). In two dimensions the assumption of [2, Thm. 5.1.9] is satisfied since $A(s)$ is a two-point set and since $\log(2/t) \to \infty$ as $t \to 0$. Thus, the $(1,2)$-capacity of $A(s)$ vanishes as well. Then, a classical approximation theorem due to Hedberg and Wolff, see [2, Thm. 9.1.3], shows that

$$\mathrm{W}^{1,2}_0(\mathbb{R}^d\setminus A(s)) = \mathrm{W}^{1,2}(\mathbb{R}^d)$$

holds, so that (5.4) holds for all $\psi \in \mathrm{W}^{1,2}(\mathbb{R}^d)$. The first consequence of this fact is that g has mean zero on $\partial[\Omega \cap U_{x_0,sr}]$, as one can take in (5.4) a function ψ that is constantly one on a neighborhood of $\Omega \cap U_{x_0,sr}$.

Next, let v be a solution to the L^2-Neumann problem, whose existence is granted by Theorem 4.1.3. In the following, we show that $v = \mathbf{u} + c$ holds in $\Omega \cap U_{x_0,sr}$ for some constant $c \in \mathbb{C}$. To do so, recall that $\Omega \cap U_{x_0,sr}$ is a bounded Lipschitz domain by Lemma 1.3.25. Let Ω_k be a sequence of smooth domains approximating Ω from inside, whose existence is ensured by Proposition 1.3.19. Then, an integration by parts reveals

$$\int_{\Omega_k} \langle \nabla v, \nabla \psi \rangle \, \mathrm{d}x = \int_{\partial\Omega_k} \langle \nu_k, \nabla v \rangle \overline{\psi} \, \mathrm{d}\sigma_k \qquad (\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)),$$

where ν_k and σ_k denote the outward unit normal and the surface measure of $\partial\Omega_k$, respectively. Note that $\nabla v \in \mathrm{L}^{\frac{2d}{d-1}}(\Omega \cap U_{x_0,sr}) \subset \mathrm{L}^2(\Omega \cap U_{x_0,sr})$ by combining Lemma 5.1.6 with Lemma 4.1.15, so that we can take the limit on the left-hand side of the equality above. Moreover, ν_k converges pointwise almost everywhere to the outward unit normal to $\partial[\Omega \cap U_{x_0,sr}]$, so that $\langle \nu_k, \nabla v \rangle$ converges non-tangentially almost everywhere to the boundary datum g. As in the proof of Lemma 1.3.21 one uses the dominated convergence theorem to deduce that one can take the limit on the right-hand side, to obtain

$$\int_{\Omega \cap U_{x_0,sr}} \langle \nabla v, \nabla \psi \rangle \, \mathrm{d}x = \int_{\Omega \cap \partial U_{x_0,sr}} \langle \nu_{U_{x_0,sr}}, \nabla \mathbf{u} \rangle \overline{\psi} \, \mathrm{d}\sigma_{U_{x_0,sr}}, \tag{5.5}$$

By density, this holds for all $\psi \in \mathrm{W}^{1,2}(\mathbb{R}^d)$. In conclusion, we find

$$\int_{\Omega \cap U_{x_0,sr}} \langle \nabla(\mathbf{u} - v), \nabla \psi \rangle \, \mathrm{d}x = 0 \qquad (\psi \in \mathrm{W}^{1,2}(\mathbb{R}^d)).$$

By smoothness of v, we have $v \in \mathrm{L}^2_{\mathrm{loc}}(\Omega \cap U_{x_0,sr})$. Moreover, we already know that $\nabla v \in \mathrm{L}^2(\Omega \cap U_{x_0,sr})$. Thus, by means of Galdi [34, Rem. II.6.2], we find $v \in \mathrm{W}^{1,2}(\Omega \cap U_{x_0,sr})$.

Next, let ψ be any $\mathrm{W}^{1,2}(\mathbb{R}^d)$-extension of $\mathbf{u} - v$ (for the existence of such an extension note that $\Omega \cap U_{x_0,sr}$ is a bounded Lipschitz domain and then consult Stein [92, Thm. VI.5]). Then, one deduces

$$\int_{\Omega \cap U_{x_0,sr}} |\nabla(\mathbf{u} - v)|^2 \, \mathrm{d}x = 0.$$

This concludes the proof. $\square$

Now, we are in the position to imitate the proof of the interior weak reverse Hölder estimates.

Proposition 5.1.9. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded or special Lipschitz domain, and let $0 < r < r_0/3$ (r_0 is set to infinity for special Lipschitz domains) and $x_0 \in \partial\Omega$. Then there exists a constant $C > 0$ depending only on d and M such that for all $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ with $f = 0$ in $\Omega \cap B(x_0, 2\alpha_1 r)$, $u := (-\Delta_N)^{-1} \operatorname{div}(f)$, and $\alpha_1 = [4 + [20d(M+1)]^2]^{1/2}$*

$$\Big(\frac{1}{r^d} \int_{\Omega \cap B(x_0, r)} |\nabla u|^{\frac{2d}{d-1}} \, \mathrm{d}x\Big)^{\frac{d-1}{2d}} \leq C \Big(\frac{1}{r^d} \int_{\Omega \cap B(x_0, \alpha_1 r)} |\nabla u|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}$$

holds.

Proof. We proceed similarly to the proof of Proposition 5.1.7, but by establishing the weak reverse Hölder estimates on the sets $U_{x_0,r}$ first.

Let r be such that $3r < r_0$. Appealing to Lemma 5.1.8, u has a representative $\mathbf{u}$ which solves for almost every $s \in [1,2]$ the L^2-Neumann problem in $U_{x_0,sr}$ with boundary data given by (5.1). Fix s suchlike and consider the rescaled function $\mathbf{u}_r : r^{-1}[\Omega \cap U_{x_0,sr}] \to \mathbb{C}$ defined by $\mathbf{u}_r(x) := \mathbf{u}(rx)$ and derive the representation

$$\begin{aligned} r^{-1}[\Omega \cap U_{x_0,sr}] &= \{r^{-1}x_0\} + r^{-1} R_{x_0}^{-1} D_{\eta_{x_0}}(sr) \\ &= \{r^{-1}x_0\} + R_{x_0}^{-1} D_{r^{-1}\eta_{x_0}(r\cdot)}(s) =: U_{x_0,s}^{\mathrm{resc},+}. \end{aligned}$$

Note that $r^{-1}\eta_{x_0}(r\cdot)$ is Lipschitz continuous with Lipschitz constant M. By virtue of Lemma 1.3.25 the sets $U_{x_0,s}^{\mathrm{resc},+}$ are Lipschitz domains. Moreover, $\mathbf{u}_r$ solves the L^2-Neumann problem in $U_{x_0,s}^{\mathrm{resc},+}$ with the rescaled boundary data

$$g_r := \begin{cases} \langle \nu_{U_{x_0,s}^{\mathrm{resc},+}}, \nabla \mathbf{u}_r \rangle, & \text{on } \partial U_{x_0,s}^{\mathrm{resc},+} \cap r^{-1}\Omega \\ 0, & \text{on } \partial U_{x_0,s}^{\mathrm{resc},+} \cap r^{-1}\partial\Omega. \end{cases}$$

Since $\mathbf{u}_r + c$ has a representation via a single layer potential for some $c \in \mathbb{C}$ by Theorem 4.1.3, we can use Lemma 4.1.15 to conclude that $\mathbf{u}_r$ solves the L^2-Neumann problem with respect to the non-tangential maximal function N_a (which is defined in (4.13)) for some $a > 0$. Due to Remarks 4.1.4, 1.3.16 (3), and 1.3.27, the respective constant in the inequality

$$\|N_a \nabla \mathbf{u}_r\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+})} \leq C \|g_r\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+})} \tag{5.6}$$

depends merely on d, M, and some dimensional constants $\widetilde{r}$ and n_0 arising from a covering of $\partial U_{x_0,s}^{\mathrm{resc},+}$ by cylinders. Note that this can be performed uniformly in s, so that C is independent of s. Using Lemma 5.1.6 together with Proposition 4.1.11, we find

$$\bigg(\int_{U_{x_0,1}^{\mathrm{resc},+}} |\nabla \mathbf{u}_r|^{\frac{2d}{d-1}} \, \mathrm{d}x \bigg)^{\frac{d-1}{d}} \leq C \int_{\partial U_{x_0,s}^{\mathrm{resc},+}} |N_a \nabla \mathbf{u}_r|^2 \, \mathrm{d}\sigma_{U_{x_0,s}^{\mathrm{resc},+}}.$$

Since g_r vanishes on $\partial U_{x_0,s}^{\mathrm{resc},+} \cap r^{-1}\partial\Omega$, this is estimated by means of (5.6) by

$$\leq C \int_{\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega} |\nabla \mathbf{u}_r|^2 \, \mathrm{d}\sigma_{U_{x_0,s}^{\mathrm{resc},+}}.$$

Note that C is independent of s since s is bounded away from zero, so that by Lemma 1.3.25 the quantities that describe $U_{x_0,s}^{\mathrm{resc},+}$ as a Lipschitz domain are uniform. Next, with a constant depending only on d and M, the surface measure $\sigma_{U_{x_0,s}^{\mathrm{resc},+}}$ is equivalent to the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. Integrate the inequality over $s \in [1,2]$ (recall that the inequality above holds for almost every $s \in [1,2]$). Then define the Lipschitz continuous function

$$x \mapsto s \quad \text{iff} \quad x \in r^{-1} U_{x_0,sr},$$

which is set to zero at $r^{-1}x_0$. This function has a Lipschitz constant depending only on d. Using this function together wih $\mathfrak{g} := |\nabla \mathbf{u}_r|^2 \chi_{U_{x_0,2}^{\mathrm{resc},+}}$ to appeal to the co-area formula, Theorem 1.3.22, we find

$$\begin{aligned} \bigg(\int_{U_{x_0,1}^{\mathrm{resc},+}} |\nabla \mathbf{u}_r|^{\frac{2d}{d-1}} \, \mathrm{d}x \bigg)^{\frac{d-1}{d}} &\leq C \int_0^2 \int_{\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega} |\nabla \mathbf{u}_r|^2 \, \mathrm{d}m_{d-1} \, \mathrm{d}s \\ &\leq C \int_{U_{x_0,2}^{\mathrm{resc},+}} |\nabla \mathbf{u}_r|^2 \, \mathrm{d}x. \end{aligned}$$

Employing the change of variables $rx = y$ proves

$$\left(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}} |\nabla u|^{\frac{2d}{d-1}} \,\mathrm{d}x\right)^{\frac{d-1}{2d}} \leq C\left(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,2r}} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

To obtain the weak reverse Hölder estimates on balls, note that

$$B(x_0, r) \subset U_{x_0,r} \quad \text{and} \quad U_{x_0,2r} \subset B(x_0, [4 + [20d(M+1)]^2]^{1/2} r).$$

This concludes the proof. □

Now, we can prove the desired L^p-boundedness of the Helmholtz projection.

Theorem 5.1.10. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded or special Lipschitz domain. Then there exists $\varepsilon > 0$ depending only on d and M such that for all p with*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the restriction of operator $\nabla(-\Delta_N)^{-1}\operatorname{div} \in \mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))$ onto $\mathrm{L}^2(\Omega;\mathbb{C}^d)\cap \mathrm{L}^p(\Omega;\mathbb{C}^d)$ extends to a bounded operator on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$.

For the same range of p's, the restriction of the Helmholtz projection $\mathbb{P}$ onto $\mathrm{L}^2(\Omega;\mathbb{C}^d)\cap\mathrm{L}^p(\Omega;\mathbb{C}^d)$ extends to a projection on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ with range $\mathrm{L}^p_\sigma(\Omega)$.

Proof. Consider the operator $T := \nabla(-\Delta_N)^{-1}\operatorname{div} \in \mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))$. This operator satisfies the weak reverse Hölder estimates proven in Propositions 5.1.7 and 5.1.9. We appeal to the self-improving property of weak reverse Hölder estimates as follows.

Step 1: Getting an ε more.

Let $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0, 3r) \subset \Omega$. Define $p := 2d/(d-1)$, let $f \in \mathrm{L}^2(\Omega;\mathbb{C}^d)$ be zero in $B(x_0, 3r)$, and define $u := Tf$.

Let $y \in B(x_0, 2r)$ and $r' > 0$ be such that $B(y, 2r') \subset B(x_0, 2r)$. It is clear that $B(y, 3r') \subset B(x_0, 3r)$, so that f vanishes also in $B(y, 3r')$. By means of Proposition 5.1.7, the weak reverse Hölder estimate

$$\left(\frac{1}{(r')^d}\int_{B(y,r')} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C\left(\frac{1}{(r')^d}\int_{B(y,2r')} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

holds. Apply Proposition 3.1.4 with $\Omega = B(x_0, 2r)$ and $q = p/2$ to conclude that there exists an $\varepsilon > 0$ depending only on d and C such that for all $y \in B(x_0, 2r)$ and $r' > 0$ with $B(y, 2r') \subset B(x_0, 2r)$

$$\left(\frac{1}{(r')^d} \int_{B(y,r')} |u|^{p+\varepsilon} \, \mathrm{d}x \right)^{\frac{1}{p+\varepsilon}} \leq C \left(\frac{1}{(r')^d} \int_{B(y,2r')} |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

To conclude the improvement for the inner estimates take $y = x_0$ and $r' = r/2$.

If $x_0 \in \partial\Omega$ and $0 < r < r_0/3$, let $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ be zero in $\Omega \cap B(x_0, 2\alpha_1 r)$, where α_1 is as in Proposition 5.1.9. Define again $u := Tf$.

Let $y \in B(x_0, 2r)$ and $r' > 0$ be such that $B(y, 2r') \subset B(x_0, 2r)$ and $\Omega \cap B(y, 2r') \neq \emptyset$. Since $\alpha_2 \geq 2$, we have $B(y, \alpha_2 r') \subset B(x_0, \alpha_2 r)$, so that f vanishes on $B(y, \alpha_2 r')$. By means of Lemma 3.2.2, we find that also

$$\left(\frac{1}{(r')^d} \int_{\Omega \cap B(y,r')} |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq C \left(\frac{1}{(r')^d} \int_{\Omega \cap B(y,2r')} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

is valid. Now, we conclude the higher integrability as in the interior case, but by extending u trivially to all of $B(x_0, \alpha_2 r)$ by zero. Since ε depends only on the constant of the weak reverse Hölder estimates and of d, the number ε is the same for all x_0 and all radii r.

Step 2: Concluding the boundedness.

By Theorem 3.1.2 the restriction of T onto $\mathrm{L}^2(\Omega; \mathbb{C}^d) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$ extends to a bounded operator on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ if

$$2 < p < \frac{2d}{d-1} + \varepsilon.$$

By duality, the same holds true for T on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for

$$\frac{2d}{d+1} - \varepsilon < p < 2.$$

This proves the first part of the theorem.

Use the representation $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta_N)^{-1} \operatorname{div}$ proven in Lemma 5.1.3 to conclude that the restriction of $\mathbb{P}$ onto $\mathrm{L}^2(\Omega; \mathbb{C}^d) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$ extends to a bounded operator $\mathbb{P}_p$ on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for p satisfying both of the conditions above. That $\mathbb{P}_p$ is a projection is clear because the identity $\mathbb{P}_p\mathbb{P}_p = \mathbb{P}_p$ holds on a dense subset, namely on $\mathrm{L}^2(\Omega;\mathbb{C}^d) \cap \mathrm{L}^p(\Omega;\mathbb{C}^d)$.

Step 3: Identification of the range of $\mathbb{P}_p$.

To identify the range of $\mathbb{P}_p$ note that any projection has a closed range and that $\mathrm{C}^\infty_{c,\sigma}(\Omega) \subset \mathcal{R}(\mathbb{P}_p)$ since this holds true for $p = 2$, see Lemma 5.1.3. It follows that $\mathrm{L}^p_\sigma(\Omega) \subset \mathcal{R}(\mathbb{P}_p)$.

To prove the other inclusion, let $f \in \mathrm{L}^p(\Omega;\mathbb{C}^d)$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\Omega;\mathbb{C}^d)$ with $f_n \to f$ in $\mathrm{L}^p(\Omega;\mathbb{C}^d)$. Since $\mathbb{P}$ extends to the bounded operator $\mathbb{P}_p$ on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$, we find that $\mathbb{P}_p f = \lim_{n\to\infty} \mathbb{P} f_n$. This implies that

$$\int_\Omega \langle \mathbb{P}_p f, v\rangle \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \langle \mathbb{P} f_n, v\rangle \,\mathrm{d}x = 0, \tag{5.7}$$

whenever $v \in G_2(\Omega) \cap G_{p'}(\Omega)$, where p' is the Hölder conjugate exponent to p, and

$$G_r(\Omega) := \{v \in \mathrm{L}^r(\Omega;\mathbb{C}^d) : \exists \pi \in \mathrm{L}^r_{\mathrm{loc}}(\Omega) \text{ with } \nabla\pi = v\} \qquad (1 < r < \infty).$$

Assume for a moment that every function $v \in G_{p'}(\Omega)$ can be approximated by a sequence $(v_n)_{n\in\mathbb{N}} \subset G_2(\Omega) \cap G_{p'}(\Omega)$ in the $\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)$-norm. In this situation, (5.7) would hold for all $v \in G_{p'}(\Omega)$ by density. An application of Galdi [34, Lem. III.2.1] then shows that $f \in \mathrm{L}^p_\sigma(\Omega)$.

To establish the approximation result, let $v \in G_{p'}(\Omega)$ and let $\pi \in \mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ be such that $\nabla\pi = v$. The function π then lies in an equivalence class of the homogeneous Sobolev space $\dot{\mathrm{W}}^{1,p'}(\Omega)$.

If Ω is bounded, then the approximation result in [34, Thm. II.7.2] shows that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty(\overline{\Omega})$ with $\nabla\varphi_n \to v$ in $\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)$ as $n \to \infty$.

If Ω is a special Lipschitz domain, we transform the special Lipschitz domain onto the upper half-space and use the corresponding approximation results for the upper half-space. Define

$$\Phi : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto (x', x_d - \eta(x')),$$

where $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ represents $\partial\Omega$. The function Φ is bi-Lipschitzian with bi-Lipschitz constant $M + 1$ and maps Ω onto the upper half-space $\mathbb{R}^d_+$. Since $\pi \in \mathrm{W}^{1,1}(K)$ for every compact set $K \subset \Omega$, a change of variables, see Theorem 1.3.4, shows that $\pi \circ \Phi$ gives rise to an equivalence class in

$\dot{\mathrm{W}}^{1,p'}(\mathbb{R}^d_+)$. Use the approximation result in [34, Thm. II.7.8] to obtain a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathrm{C}_c^\infty(\overline{\mathbb{R}^d_+})$ with $\nabla\psi_n \to \nabla[\pi \circ \Phi]$ in $\mathrm{L}^{p'}(\mathbb{R}^d_+;\mathbb{C}^d)$ as $n \to \infty$. Define $\varphi_n := \psi_n \circ \Phi^{-1}$ and calculate by using the chain rule and change of variables

$$\begin{aligned}\int_\Omega &|\partial_j \varphi_n(x) - \partial_j \pi(x)|^{p'} \,\mathrm{d}x \\ &= \int_\Omega \Big|\Big\langle [\nabla\psi_n](\Phi^{-1}(x)) - [\nabla\pi \circ \Phi](\Phi^{-1}(x)), \partial_j \Phi^{-1}(x)\Big\rangle\Big|^{p'} \,\mathrm{d}x \\ &\leq (M+1)^{p'} \int_{\mathbb{R}^d_+} |[\nabla\psi_n](y) - [\nabla\pi\circ\Phi](y)|^{p'} \,\mathrm{d}y,\end{aligned}$$

which converges to zero as $n \to \infty$. □

Convention 5.1.11. For the rest of this thesis, we refer to the Helmholtz projection on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ by $\mathbb{P}_p$, even if $p = 2$.

5.2 The Stokes operator and semigroup

The Stokes operator and the Stokes semigroup are one of the central objects in the study of incompressible fluid flow. Especially, they are fundamental for the approaches of Kato and Fujita [60], Kato [59], and Giga [41]. For bounded and smooth domains, the articles of Solonnikov [91], Giga [39], and of Giga and Miyakawa [42] lay the foundation of the semigroup theory and the study of the domains of fractional powers of the Stokes operator if the underlying domain is smooth.

On bounded Lipschitz domains, there are only recent developments in the study of the Stokes operator and the Stokes semigroup. The articles of Brown and Shen [13], Mitrea and Monniaux [75] and Mitrea, Monniaux, and Wright [76] give a clear and good impression of the situation in $\mathrm{L}^2_\sigma(\Omega)$ under Dirichlet and Neumann boundary conditions. Furthermore, Mitrea and Wright [77] establish mapping properties of the Stokes operator in the scale of Besov and Triebel-Lizorkin spaces. Thereby, they make it possible to obtain embeddings of the domain of the Stokes operator, when considered as an operator on $\mathrm{L}^p_\sigma(\Omega)$ with p in a certain interval about 2. In this context, we should also mention the work of Geng and Kilty [36], who establish L^p-estimates of the gradient of the solution to the Stokes equations with right-hand sides in divergence form.

The breakthrough in the study of the Stokes semigroup on $\mathrm{L}^p_\sigma(\Omega)$ was made by Shen [89] by showing that the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ generates a bounded analytic semigroup, again with p in a certain interval around 2.

In this section, we will introduce the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ as proposed in [75] and will establish some of the elementary properties of the Stokes operator. We will collect known results and prove easy properties, whose proofs could not be found by the author in the literature. In the end of this section, we will prove that the Stokes operator has maximal L^q-regularity and establish various types of L^p-L^q-estimates of the Stokes semigroup. These two results unkown until the present moment and open up the door to develop an L^p-theory of the Navier-Stokes equations.

5.2.1 The Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$

We start with the definition of the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ via the standard form method.

Definition 5.2.1. Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be any open set. Define the sesquilinear form

$$\mathfrak{a} : \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \times \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \langle \nabla u, \nabla v \rangle \, \mathrm{d}x.$$

The *Stokes operator* on $\mathrm{L}^2_\sigma(\Omega)$ is the operator A_2, whose domain $\mathcal{D}(A_2)$ is given by

$$\Big\{ u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \exists f \in \mathrm{L}^2_\sigma(\Omega) \text{ s.t. } \forall v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \mathfrak{a}(u, v) = \int_\Omega \langle f, v \rangle \, \mathrm{d}x \Big\}.$$

With f belonging to $u \in \mathcal{D}(A_2)$ via the relationship given in the definition of $\mathcal{D}(A_2)$, we define

$$A_2 u := f.$$

Remark 5.2.2. By classical form theory, see Kato [58], it is clear that A_2 is a densely defined and closed operator. By the symmetry of $\mathfrak{a}$, A_2 is self-adjoint and since additionally

$$\mathfrak{a}(u, u) = \|\nabla u\|^2_{\mathrm{L}^2(\Omega; \mathbb{C}^{d^2})} \geq 0 \qquad (u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)),$$

we find that the spectrum lies on the non-negative real axis.

On bounded Lipschitz domains Ω, there is the following characterization of $\mathcal{D}(A_2)$, see [75, Thm. 4.7].

Theorem 5.2.3. *If $\Omega \subset \mathbb{R}^d$, $d \geq 3$, is a bounded Lipschitz domain, then the domain of the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ is given by*

$$\{u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \exists \pi \in \mathrm{L}^2(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^2_\sigma(\Omega)\},$$

where $-\Delta u + \nabla \pi \in \mathrm{L}^2_\sigma(\Omega)$ is understood in the sense of distributions. If π is the pressure belonging to $u \in \mathcal{D}(A_2)$, then $A_2 u$ is given by

$$A_2 u = -\Delta u + \nabla \pi.$$

To establish resolvent estimates in $\mathrm{L}^2_\sigma(\Omega)$ the following lemma is important. Note that we state it in a far more general form as it is needed in this chapter. However, this generality allows us to use it in Chapter 7 again.

Lemma 5.2.4. *Let $\theta_1, \theta_2 \in [0, \pi)$ with $\theta := \theta_1 + \theta_2 < \pi$. Then, there exists a constant C_θ, depending only on θ, such that for all $A \geq 0$, $\lambda \in \mathrm{S}_{\theta_1}$, and $B \in \mathrm{S}_{\theta_2}$ the inequality*

$$|A\lambda + B| \geq C_\theta \{A\,|\lambda| + |B|\}$$

holds.

Proof. Note that $\lambda\overline{B}$ lies in the sector S_θ. This shows that

$$\begin{aligned} |\lambda A + B|^2 &= |\lambda|^2 A^2 + 2A \operatorname{Re}(\lambda \overline{B}) + |B|^2 \\ &\geq |\lambda|^2 A^2 + 2A\,|B|\,|\lambda| \cos(\theta) + |B|^2 . \end{aligned}$$

If $\theta \leq \pi/2$, then the right-hand side is greater or equal than $|\lambda|^2 A^2 + |B|^2$. If $\theta \in (\pi/2, \pi)$, use $2A\,|B|\,|\lambda| \leq |\lambda|^2 A^2 + B^2$ to derive that the right-hand side is greater or equal than

$$(1 + \cos(\theta))|\lambda|^2 A^2 + (1 + \cos(\theta))|B|^2.$$

The lemma follows by means of the inequality

$$\Big[\,|\lambda|^2 A^2 + |B|^2\Big]^{1/2} \geq \frac{|\lambda|\,A + |B|}{\sqrt{2}}. \qquad \square$$

For a function $F \in \mathrm{W}^{1,p}(\Omega; \mathbb{C}^{d\times d})$ define its divergence as the vector

$$\operatorname{div}(F) := \begin{pmatrix} \sum_{i=1}^d \partial_i F_{i1} \\ \vdots \\ \sum_{i=1}^d \partial_i F_{id} \end{pmatrix}. \tag{5.8}$$

Next, we present the classical proof of the resolvent and the gradient estimates of A_2.

Proposition 5.2.5. *Let Ω be open and $\theta \in [0, \pi)$. Then there exists a constant $C > 0$ depending only on θ, such that for all $\lambda \in \mathrm{S}_\theta$ and $f \in \mathrm{L}^2_\sigma(\Omega)$*

$$\begin{aligned} |\lambda| \, \|(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} &\leq C\|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \\ |\lambda|^{1/2} \, \|\nabla(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})} &\leq C\|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}. \end{aligned}$$

If Ω is bounded, then A_2 is invertible and the inverse is bounded from $\mathrm{L}^2_\sigma(\Omega)$ into $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$.

Moreover, for all $\lambda \in \mathrm{S}_\theta$ the operator

$$(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div} : \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^{d\times d}) \to \mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d)$$

extends to a bounded operator on all of $\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$, which satisfies for all $F \in \mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$

$$\begin{aligned} |\lambda|^{1/2} \, \|(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} &\leq C\|F\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d\times d})} \\ \|\nabla(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} &\leq C\|F\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d\times d})} \end{aligned}$$

where C is the same constant as above.

If Ω is bounded, then $A_2^{-1}\mathbb{P}_2 \operatorname{div}$ extends to a bounded operator from $\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$ into $\mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d)$.

Proof. Using Lemma 5.2.4, we see that for all $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$

$$\Big|\lambda \int_\Omega |u|^2 \, \mathrm{d}x + \mathfrak{a}(u,u)\Big| \geq C_\theta \Big\{ |\lambda| \, \|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})} \Big\}. \tag{5.9}$$

Consequently, by the lemma of Lax–Milgram, for any $\lambda \in \mathrm{S}_\theta$ and $f \in \mathrm{L}^2_\sigma(\Omega)$ we find a unique $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ such that

$$\lambda \int_\Omega \langle u, v \rangle \, \mathrm{d}x + \mathfrak{a}(u, v) = \int_\Omega \langle f, v \rangle \, \mathrm{d}x \qquad (v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d)).$$

Plugging u into this equality, an application of (5.9) and Hölder's inequality yield

$$|\lambda| \, \|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})} \leq C_\theta^{-1} \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}.$$

Forgetting the term involving the gradient on the left-hand side yields the first inequality.

For the second inequality, use Young's inequality to derive

$$C_\theta^{-1} \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq \frac{1}{4 C_\theta^2 \, |\lambda|} \|f\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} + |\lambda| \, \|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}$$

and cancel the terms involving the L^2-norm of u in the inequality before.

If $f = \mathbb{P}_2 \operatorname{div}(F)$ for some $F \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^{d \times d})$, we find by the self-adjointness of $\mathbb{P}_2$ and by integration by parts

$$\begin{aligned} \int_\Omega \langle \mathbb{P}_2 \operatorname{div}(F), v \rangle \, \mathrm{d}x &= \sum_{i,j=1}^d \int_\Omega \partial_i F_{ij} \overline{v_j} \, \mathrm{d}x \\ &= - \sum_{i,j=1}^d \int_\Omega F_{ij} \overline{\partial_i v_j} \, \mathrm{d}x \qquad (v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d)). \end{aligned}$$

Thus, similarly as above, we derive

$$|\lambda| \, \|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})} \leq C_\theta^{-1} \|F\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d \times d})} \|\nabla u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})}.$$

The estimate on ∇u follows simply by forgetting the L^2-norm of u on the left-hand side. An application of Young's inequality

$$C_\theta^{-1} \|F\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d \times d})} \|\nabla u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})} \leq \frac{1}{4 C_\theta^2} \|F\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{d \times d})} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})}$$

delivers the estimate on u.

Finally, the statements for bounded domains follow by an application of Poincaré's inequality

$$\mathfrak{a}(u, u) \geq C \|u\|^2_{\mathrm{W}^{1,2}(\Omega;\mathbb{C}^d)} \qquad (u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

with a positive constant C and by the Lax–Milgram lemma. $\square$

Now, that we know that A_2 is sectorial of any angle $\omega \in [0, \pi)$, we can consider fractional powers of the Stokes operator. The following characterization of the domains $\mathcal{D}(A_2^s)$ with $0 < s < 3/4$, $s \neq 1/4$, was crucial in [75] and will be similarly important in this treatise. For a proof of the theorem, see [75, Cor. 5.2 & Thm. 5.3].

Theorem 5.2.6 (Mitrea, Monniaux). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then for each $0 < s < 3/4$, there exists a constant $C > 0$ such that*

$$\|u\|_{\mathrm{H}^{2s,2}(\Omega;\mathbb{C}^d)} \leq C \|A_2^s u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (u \in \mathcal{D}(A_2^s)).$$

Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{\mathrm{H}^{3/2,2}(\Omega;\mathbb{C}^d)} \leq C \|u\|_{\mathcal{D}(A_2)} \qquad (u \in \mathcal{D}(A_2)).$$

The previous results describe the situation in $\mathrm{L}^2_\sigma(\Omega)$ very well and we will turn to investigate the situation in $\mathrm{L}^p_\sigma(\Omega)$ for $p \neq 2$.

5.2.2 The Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$

We start with the definition of the Stokes operator for $p > 2$.

Definition 5.2.7. For $p > 2$, define the L^p_σ-realization A_p of A_2 as the *part* of A_2 in $\mathrm{L}^p_\sigma(\Omega)$, i.e.,

$$\begin{aligned} \mathcal{D}(A_p) &:= \{u \in \mathcal{D}(A_2) : u \in \mathrm{L}^p_\sigma(\Omega) \text{ and } A_2 u \in \mathrm{L}^p_\sigma(\Omega)\} \\ A_p u &:= A_2 u. \end{aligned}$$

To prove that $A_2^{-1} f$ is in $\mathcal{D}(A_p)$ for some $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$, one usually knows that for example $A_2^{-1} f \in \mathrm{L}^2_\sigma(\Omega)$ and that additionally an integration property holds, like $A_2^{-1} f \in \mathrm{L}^p(\Omega; \mathbb{C}^d)$. But, to conclude that $A_2^{-1} f \in \mathcal{D}(A_p)$, we need to know that $A_2^{-1} f \in \mathrm{L}^p_\sigma(\Omega)$ holds true. That means, we need to know whether $\mathrm{L}^2_\sigma(\Omega) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$ coincides with $\mathrm{L}^p_\sigma(\Omega)$. While this follows from a characterization of $\mathrm{L}^p_\sigma(\Omega)$ on bounded Lipschitz domains, the author could not find a respective result for first-order Sobolev spaces in the literature. We care about this consistency question in the following technical lemma.

Lemma 5.2.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and $1 < q < p$. Then*

$$\mathrm{L}^q_\sigma(\Omega) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d) = \mathrm{L}^p_\sigma(\Omega)$$
$$\mathrm{W}^{1,q}_{0,\sigma}(\Omega) \cap \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d) = \mathrm{W}^{1,p}_{0,\sigma}(\Omega).$$

Proof. The first statement follows from Galdi [34, Thm. III.2.3].

For the second statement, note that the inclusion

$$\mathrm{W}^{1,q}_{0,\sigma}(\Omega) \cap \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d) \supset \mathrm{W}^{1,p}_{0,\sigma}(\Omega)$$

follows by Hölder's inequality. To prove the other inclusion, note that by (1.4)

$$\mathrm{W}^{1,r}_{0,\sigma}(\Omega) = \{u \in \mathrm{W}^{1,r}_0(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\} \qquad (1 < r < \infty),$$

Moreover, appealing to the remark below [49, Cor. 3.5] of Hedberg and Kilpeläinen, we have

$$\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d) \cap \mathrm{W}^{1,1}_0(\Omega; \mathbb{C}^d) = \mathrm{W}^{1,p}_0(\Omega; \mathbb{C}^d).$$

Combining the two equalities above reveals

$$\begin{aligned} \mathrm{W}^{1,q}_{0,\sigma}(\Omega) \cap \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d) &= \{u \in \mathrm{W}^{1,q}_0(\Omega; \mathbb{C}^d) \cap \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\} \\ &\subset \{u \in \mathrm{W}^{1,p}_0(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\} \\ &= \mathrm{W}^{1,p}_{0,\sigma}(\Omega). \end{aligned}$$ □

Amongst other things, the following result gives embeddings of $\mathcal{D}(A_p)$ into certain Bessel potential spaces. This is a special case of the results of Mitrea and Wright [77, Thm. 10.15]. Note that some of these embeddings were established with shorter proofs by Brown and Shen [13, Thm. 2.9] for $d = 3$ and by Geng and Kilty [36, Thm. 1.3] for $d \geq 4$, and that Geng and Kilty assumed that Ω has a connected boundary.

Theorem 5.2.9. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all $2 < p < 2d/(d-1) + \varepsilon$*

(1) the operator A_p is closed;

(2) $0 \in \rho(A_p)$ *and* $A_2^{-1}|_{\mathrm{L}^p_\sigma(\Omega)} = A_p^{-1}$*;*

(3) $A_p^{-1}\mathbb{P}_p \operatorname{div}$ *is bounded from* $\mathrm{L}^p(\Omega; \mathbb{C}^{d\times d})$ *into* $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$*;*

(4) for every $s \geq 0$ *with*

$$s < \frac{3}{2} - \frac{p-2}{2p} \frac{2d + \varepsilon(d-1)}{2d + (\varepsilon - 2)(d-1)}$$

the continuous inclusion

$$\mathcal{D}(A_p) \subset \mathrm{H}^{s,p}(\Omega; \mathbb{C}^d)$$

holds. Especially, we have $\mathcal{D}(A_p) \subset \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$*.*

Proof. We begin with (2). Since A_2 is injective, it is clear that A_p is injective as well. Next, [77, Thm. 10.15] shows that for the given range of p's, $A_2^{-1} f \in \mathrm{L}^p(\Omega; \mathbb{C}^d)$ if $f \in \mathrm{L}^p_\sigma(\Omega)$. By Lemma 5.2.8, we conclude that $A_2^{-1} f \in \mathrm{L}^p_\sigma(\Omega)$ so that $A_2^{-1} f \in \mathcal{D}(A_p)$ for every $f \in \mathrm{L}^p_\sigma(\Omega)$. This implies the surjectivity of A_p so that A_p is bijective. The estimate in [77, Thm. 10.15] gives that A_p^{-1} is bounded. We derive that $0 \in \rho(A_p)$ and that $A_2^{-1}|_{\mathrm{L}^p_\sigma(\Omega)} = A_p^{-1}$.

Now, we derive (1) from (2), since bounded operators defined on the whole Banach space are always closed and since closedness is preserved by taking the inverse.

That $A_p^{-1}\mathbb{P}_p \operatorname{div}$ is bounded from $\mathrm{L}^p(\Omega; \mathbb{C}^{d\times d})$ into $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$ follows directly from the estimates given in [77, Thm. 10.15].

To prove (4), note that the operator $A_2^{-1}\mathbb{P}_2$ is bounded from $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ into $\mathrm{H}^{3/2,2}(\Omega; \mathbb{C}^d)$ by Theorem 5.2.6. Moreover, for every q with $2 < q < 2d/(d-1) + \varepsilon$ in the range in the statement above $A_q^{-1}\mathbb{P}_q$ is a bounded operator from $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ into $\mathrm{W}^{1,q}(\Omega; \mathbb{C}^d)$ by Theorem 5.1.10 and [77, Thm. 10.15]. For $\delta > 0$ small enough, we find that

$$p < \frac{2d}{d-1} + \varepsilon - \delta =: p_\delta.$$

Use the complex interpolation theorem, Theorem 1.2.4, to conclude that $A_p^{-1}\mathbb{P}_p$ is bounded from $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ into $\mathrm{H}^{s_\delta,p}(\Omega; \mathbb{C}^d)$ with

$$s_\delta = \frac{3}{2} - \frac{p-2}{2p} \frac{2d + (\varepsilon - \delta)(d-1)}{2d + (\varepsilon - \delta - 2)(d-1)}.$$

For $\delta \to 0$ the number s_δ becomes the number s given in the theorem. $\square$

Remark 5.2.10. The closedness of A_p was also proven by Shen in [89, Rem. 6.4] by different means.

A consequence of the previous theorem is the following analog of Theorem 5.2.3 in the L^p-setting. Note that in [89], Shen actually defined the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ directly by virtue of this representation.

Theorem 5.2.11. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p is given by

$$\mathcal{D}(A_p) = \{u \in \mathrm{W}^{1,p}_{0,\sigma}(\Omega) : \exists \pi \in \mathrm{L}^p(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)\},$$

where $-\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)$ is understood in the sense of distributions. If π is the pressure belonging to $u \in \mathcal{D}(A_p)$, then $A_p u$ is given by

$$A_p u = -\Delta u + \nabla \pi.$$

Proof. Let $u \in \mathcal{D}(A_p)$. By Theorem 5.2.9, we find $u = A_p^{-1} f$ for some $f \in \mathrm{L}^p_\sigma(\Omega)$ and $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$. Since additionally $\mathrm{L}^p_\sigma(\Omega) \subset \mathrm{L}^2_\sigma(\Omega)$, we have $u = A_2^{-1} f$ so that $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$. By Lemma 5.2.8, we find $u \in \mathrm{W}^{1,p}_{0,\sigma}(\Omega)$. Since u is particularly in $\mathcal{D}(A_2)$, there exists $\pi \in \mathrm{L}^2(\Omega)$ with

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, \mathrm{d}x - \int_\Omega \pi \operatorname{div}(\varphi) \, \mathrm{d}x = \int_\Omega \langle f, \varphi \rangle \, \mathrm{d}x \qquad (\varphi \in \mathrm{C}^\infty_c(\Omega; \mathbb{C}^d)).$$

By density the equality holds for all $\varphi \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^d)$. Subtracting a constant from π, we may assume that the mean value of π is zero.

Let $\mathcal{B}$ be Bogovskiĭ's operator, i.e., for each $1 < r < \infty$, $\mathcal{B}$ is the operator that maps $\mathrm{L}^r_0(\Omega)$ boundedly into $\mathrm{W}^{1,r}_0(\Omega; \mathbb{C}^d)$ and that satisfies

$$\operatorname{div}(\mathcal{B} g) = g \qquad (g \in \mathrm{L}^r_0(\Omega)),$$

see GALDI [34, Sec. III.3], or BOGOVSKIĬ [11] for a construction. Then, since $\mathrm{L}^2_0(\Omega)$ is dense in $\mathrm{L}^{p'}_0(\Omega)$,

$$\begin{aligned}
\|\pi\|_{\mathrm{L}^p(\Omega)} &= \sup_{\substack{g\in \mathrm{L}^2_0(\Omega)\\ \|g\|_{\mathrm{L}^{p'}(\Omega)}\leq 1}} \Big| \int_\Omega \pi \overline{g} \, \mathrm{d}x \Big| \\
&= \sup_{\substack{g\in \mathrm{L}^2_0(\Omega)\\ \|g\|_{\mathrm{L}^{p'}(\Omega)}\leq 1}} \Big| \int_\Omega \pi \operatorname{div}(\overline{\mathcal{B}g}) \, \mathrm{d}x \Big| \\
&\leq \sup_{\substack{g\in \mathrm{L}^2_0(\Omega)\\ \|g\|_{\mathrm{L}^{p'}(\Omega)}\leq 1}} \Big\{ \int_\Omega |\nabla u| \, |\nabla \mathcal{B} g| \; \mathrm{d}x + \int_\Omega |f| \, |\mathcal{B} g| \; \mathrm{d}x \Big\}.
\end{aligned}$$

Finally, Hölder's inequality and the boundedness of Bogovskiĭ's operator yield

$$\leq \|\mathcal{B}\|_{\mathcal{L}(\mathrm{L}^{p'}_0(\Omega), \mathrm{W}^{1,p'}_0(\Omega;\mathbb{C}^d))} \Big\{ \|\nabla u\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} + \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \Big\}.$$

To prove the other inclusion, denote the set on the right-hand side of the theorem by $\mathcal{D}$. Since Ω is bounded and $u \in \mathcal{D}$, we readily see by means of Hölder's inequality and Theorem 5.2.3 that $u \in \mathcal{D}(A_2)$. Moreover, by assumption and Theorem 5.2.3, $u \in \mathrm{L}^p_\sigma(\Omega)$ and $A_2 u = -\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)$. By the definition of $\mathcal{D}(A_p)$, we conclude that $u \in \mathcal{D}(A_p)$. □

A corollary of this result is that the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ is densely defined.

Corollary 5.2.12. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$2 \leq p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p is dense in $\mathrm{L}^p_\sigma(\Omega)$. More precisely, $\mathrm{C}^\infty_{c,\sigma}(\Omega) \subset \mathcal{D}(A_p)$ and it holds

$$A_p u = -\mathbb{P}_p \Delta u \qquad (u \in \mathrm{C}^\infty_{c,\sigma}(\Omega)).$$

Proof. For p in the range above, the Helmholtz projection $\mathbb{P}_p$ is bounded on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ due to Theorem 5.1.10. Recall the representation of $\mathbb{P}_2$ given by Lemma 5.1.3 and that $\mathbb{P}_p$ is the restriction of $\mathbb{P}_2$ onto $\mathrm{L}^p(\Omega;\mathbb{C}^d)$. Then, for $\varphi \in \mathrm{C}^\infty_{c,\sigma}(\Omega)$,

$$-\Delta\varphi = -\mathbb{P}_p\Delta\varphi + (\mathbb{P}_p - \mathrm{Id})\Delta\varphi = -\mathbb{P}_p\Delta\varphi + \nabla(-\Delta_N)^{-1}\operatorname{div}(\Delta\varphi),$$

what shows that $-\Delta\varphi + \nabla(-\Delta_N)^{-1}\operatorname{div}(-\Delta\varphi) \in \mathrm{L}^p_\sigma(\Omega)$. Since $\varphi \in \mathrm{C}^\infty_c(\Omega)$, it follows that $(-\Delta_N)^{-1}\operatorname{div}(-\Delta\varphi) \in \dot{\mathrm{W}}^{1,2}(\Omega)$ by definition of this operator. Hence, any representative of $(-\Delta_N)^{-1}\operatorname{div}(-\Delta\varphi)$ lies in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Since $\nabla(-\Delta_N)^{-1}\operatorname{div}(-\Delta\varphi) \in \mathrm{L}^p(\Omega;\mathbb{C}^d)$ by Theorem 5.1.10, we conclude that $(-\Delta_N)^{-1}\operatorname{div}(-\Delta\varphi) \in \mathrm{W}^{1,p}(\Omega)$ by Galdi [34, Rem. II.6.2]. Theorems 5.2.3 and 5.2.11 show that $\varphi \in \mathcal{D}(A_p)$ and that $A_p\varphi = -\mathbb{P}_p\Delta\varphi$. By density of $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ in $\mathrm{L}^p_\sigma(\Omega)$, we conclude the proof. □

In order to consider the situation for $p < 2$, we would like to define A_p by dualizing $A_{p'}$. For this purpose, we have to start with a technical lemma.

Lemma 5.2.13. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the spaces $\mathrm{L}^p_\sigma(\Omega)$ and $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ are isomorphic, where $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ denotes the space of antilinear, bounded mappings from $\mathrm{L}^{p'}_\sigma(\Omega)$ into $\mathbb{C}$ and where p' is the Hölder conjugate exponent of p. Moreover, the isomorphism Φ is given by

$$[\Phi f](g) = \int_\Omega \langle f, g\rangle \,\mathrm{d}x \qquad (g \in \mathrm{L}^{p'}_\sigma(\Omega)).$$

Proof. Let $f \in \mathrm{L}^p_\sigma(\Omega)$. Then Φf is antilinear and by means of Hölder's inequality, it satisfies

$$|[\Phi f](g)| \leq \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}\|g\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)} \qquad (g \in \mathrm{L}^{p'}_\sigma(\Omega)).$$

It follows that $\Phi f \in (\mathrm{L}^{p'}_\sigma(\Omega))^*$ and $\|\Phi\|_{\mathcal{L}(\mathrm{L}^p_\sigma(\Omega),(\mathrm{L}^{p'}_\sigma(\Omega))^*)} \leq 1$.

For the other inclusion, let $h \in (\mathrm{L}^{p'}_\sigma(\Omega))^*$. Because the Helmholtz projection $\mathbb{P}_{p'}$ has range $\mathrm{L}^{p'}_\sigma(\Omega)$ by Theorem 5.1.10, we have

$$h(g) = h(\mathbb{P}_{p'} g) \qquad (g \in \mathrm{L}^{p'}_\sigma(\Omega)).$$

Define an extension $H \in (\mathrm{L}^{p'}(\Omega; \mathbb{C}^d))^*$ of h by setting

$$H(g) := h(\mathbb{P}_{p'} g) \qquad (g \in \mathrm{L}^{p'}(\Omega; \mathbb{C}^d)).$$

By means of RUDIN [83, Thm. 6.16], there exists a function $f \in \mathrm{L}^p(\Omega; \mathbb{C}^d)$ with $\|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} = \|H\|_{(\mathrm{L}^{p'}(\Omega;\mathbb{C}^d))^*}$ such that

$$H(g) = \int_\Omega \langle f, g \rangle \, \mathrm{d}x \qquad (g \in \mathrm{L}^{p'}(\Omega; \mathbb{C}^d)).$$

It follows that $\|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq \|\mathbb{P}_{p'}\|_{\mathcal{L}(\mathrm{L}^{p'}(\Omega;\mathbb{C}^d))} \|h\|_{(\mathrm{L}^{p'}_\sigma(\Omega))^*}$. Note that we can conclude the proof, once $f \in \mathrm{L}^p_\sigma(\Omega)$. To do so, let $g \in G_{p'}(\Omega) \cap G_2(\Omega)$, where $G_r(\Omega)$ was defined in Step 3 of the proof of Theorem 5.1.10 for $1 < r < \infty$. Then,

$$H(g) = h(\mathbb{P}_{p'} g) = h(\mathbb{P}_2 g) = 0,$$

since $\mathbb{P}_2$ is an orthogonal projection by its very definition, and since $g \in \mathrm{L}^2_\sigma(\Omega)^\perp$ by GALDI [34, Lem. III.2.1]. Since $G_{p'}(\Omega) \cap G_2(\Omega)$ is dense in $G_{p'}(\Omega)$ by [34, Thm. II.7.2], we have

$$H(g) = \int_\Omega \langle f, g \rangle \, \mathrm{d}x = 0 \qquad (g \in G_{p'}(\Omega)).$$

Now, [34, Lem. III.2.1] implies that $f \in \mathrm{L}^p_\sigma(\Omega)$. $\square$

We proceed by defining the Stokes operator for $p < 2$.

Definition 5.2.14. Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and let $\varepsilon > 0$ be the minimum of the ε's appearing in Theorem 5.1.10 and Theorem 5.2.9. Let

$$\frac{2d}{d+1} - \varepsilon < p < 2$$

and let $\Phi : \mathrm{L}^p_\sigma(\Omega) \to (\mathrm{L}^{p'}_\sigma(\Omega))^*$ be the canonical isomorphism between $\mathrm{L}^p_\sigma(\Omega)$ and the antidual of $\mathrm{L}^{p'}_\sigma(\Omega)$. Then, define the *Stokes operator on* $\mathrm{L}^p_\sigma(\Omega)$ to be

$$\begin{aligned}\mathcal{D}(A_p) &:= \{u \in \mathrm{L}^p_\sigma(\Omega) : \Phi u \in \mathcal{D}(A'_{p'})\} \\ A_p u &:= \Phi^{-1} A'_{p'} \Phi u,\end{aligned}$$

where p' denotes the Hölder conjugate exponent of p and $A'_{p'}$ the adjoint of $A_{p'}$.

Remark 5.2.15. We will use the notation Φ for the canonical isomorphism independendly of p.

By Definition 5.2.14 it is a priori not clear how much A_p has to do with A_2. This is clarified by the following proposition.

Proposition 5.2.16. *In the configuration of Definition 5.2.14, the operator A_2 is closable in $\mathrm{L}^p_\sigma(\Omega)$ and A_p is the closure of A_2 in $\mathrm{L}^p_\sigma(\Omega)$.*

Proof. To show that A_2 is closable, take $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A_2)$ with $u_n \to 0$ in $\mathrm{L}^p_\sigma(\Omega)$ and $A_2 u_n \to f$ in $\mathrm{L}^p_\sigma(\Omega)$. Then, by the self-adjointness of A_2, we derive

$$\int_\Omega \langle f, v\rangle \, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega \langle u_n, A_2 v\rangle \, \mathrm{d}x = 0 \qquad (v \in \mathcal{D}(A_{p'})).$$

The density of $\mathcal{D}(A_{p'})$ in $\mathrm{L}^{p'}_\sigma(\Omega)$ then implies $f = 0$. Hence, A_2 is closable.

Denote the closure of A_2 in $\mathrm{L}^p_\sigma(\Omega)$ by $\overline{A_2}^p$ and let $u \in \mathcal{D}(\overline{A_2}^p)$. By definition of the closure, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A_2)$ such that $u_n \to u$ and $A_2 u_n \to \overline{A_2}^p u$ in $\mathrm{L}^p_\sigma(\Omega)$. This implies for $v \in \mathcal{D}(A_{p'})$

$$\begin{aligned}[\Phi\overline{A_2}^p](v) = \int_\Omega \langle \overline{A_2}^p u, v\rangle \, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega \langle u_n, A_2 v\rangle \, \mathrm{d}x &= \int_\Omega \langle u, A_{p'} v\rangle \, \mathrm{d}x \\ &= [\Phi u](A_{p'} v).\end{aligned}$$

As a consequence, $\Phi u \in \mathcal{D}(A'_{p'})$ and $\overline{A_2}^p u = A_p u$.

To show equality, note that $A'_{p'}$ is injective, as $A_{p'}$ is surjective by standard annihilator relations, see Schechter [85, Thm. 7.15 & Thm. 7.16]. Since Φ is an isomorphism, we conclude that A_p is injective.

Next, we show that $\overline{A_2}^p$ is surjective. By density, for any given $f \in \mathrm{L}^p_\sigma(\Omega)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2_\sigma(\Omega)$ with $f_n \to f$ in $\mathrm{L}^p_\sigma(\Omega)$ as $n \to \infty$. Define $u_n := A_2^{-1} f_n$ and note that

$$
\begin{aligned}
\|u_n - u_m\|_{\mathrm{L}^p_\sigma(\Omega)} &= \sup_{\substack{g\in \mathrm{L}^{p'}_\sigma(\Omega) \\ \|g\|_{\mathrm{L}^{p'}_\sigma(\Omega)}\leq 1}} \Big| \int_\Omega \langle u_n - u_m, g\rangle \, \mathrm{d}x \Big| \\
&= \sup_{\substack{g\in \mathrm{L}^{p'}_\sigma(\Omega) \\ \|g\|_{\mathrm{L}^{p'}_\sigma(\Omega)}\leq 1}} \Big| \int_\Omega \langle f_n - f_m, A_{p'}^{-1} g\rangle \, \mathrm{d}x \Big| \\
&\leq \|A_{p'}^{-1}\|_{\mathcal{L}(\mathrm{L}^{p'}_\sigma(\Omega))} \|f_n - f_m\|_{\mathrm{L}^p_\sigma(\Omega)}.
\end{aligned}
$$

We derive that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p_\sigma(\Omega)$, so that its limit u lies in $\mathcal{D}(\overline{A_2}^p)$ and $\overline{A_2}^p u = f$. Summarizing, A_p and $\overline{A_2}^p$ coincide on $\mathcal{D}(\overline{A_2}^p)$, $\overline{A_2}^p$ is surjective, and A_p is injective. In this configuration, both operators must already coincide. $\square$

The following theorem is an analog of Theorem 5.2.9 for $p < 2$.

Theorem 5.2.17. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all $2d/(d+1) - \varepsilon < p < 2$*

(1) the operator A_p is closed and densely defined;

(2) $0 \in \rho(A_p)$;

(3) A_p^{-1} is bounded from $\mathrm{L}^p_\sigma(\Omega)$ into $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$;

(4) for every $0 \leq s < 1 + 1/p$ the continuous inclusion

$$\mathcal{D}(A_p) \subset \mathrm{H}^{s,p}(\Omega; \mathbb{C}^d)$$

holds.

Proof. The closedness of A_p follows by Proposition 5.2.16. The same proposition implies that $\mathcal{D}(A_2) \subset \mathcal{D}(A_p)$ and by Corollary 5.2.12 that this is dense in $\mathrm{L}^p_\sigma(\Omega)$.

To derive (2), let p' be the Hölder conjugate exponent of p. By Theorem 5.2.9, we know that $0 \in \rho(A_{p'})$ and that $A_{p'}$ is closed. Moreover, $A_{p'}$ is densely defined by Corollary 5.2.12. Appealing to Schechter [85, Thm. 7.15 & Thm. 7.16], we conclude that $0 \in \rho(A'_{p'})$. Since Φ is an isomorphism and $A_p = \Phi^{-1} A'_{p'} \Phi$, the same holds true for A_p.

Finally, note that (3) and (4) are special cases of the results of MITREA and WRIGHT [77, Thm. 10.15]. □

So far, we only know some spectral theory of A_p if $p = 2$. The following proposition shows that the spectrum consists only of countably many eigenvalues and that the spectrum of A_p is independent of p.

Proposition 5.2.18. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and let $\varepsilon > 0$ be the minimum of the ε's appearing in Theorem 5.1.10 and Theorem 5.2.9. Let*

$$\frac{2d}{d+1} - \varepsilon < p \leq q < \frac{2d}{d-1} + \varepsilon,$$

then the spectrum of A_p is independent of p and consists solely of countably many positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, which converge to infinity as $n \to \infty$.

For all $\lambda \in \rho(A_2)$, we have $(\lambda - A_p)^{-1}|_{\mathrm{L}^q_\sigma(\Omega)} = (\lambda - A_q)^{-1}$.

Proof. First, note that A_2^{-1} is bounded from $\mathrm{L}^2_\sigma(\Omega)$ into $\mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d)$ by Proposition 5.2.5. The theorem of Rellich and Kondrachov implies that the embedding $\mathrm{W}^{1,2}_{0,\sigma}(\Omega; \mathbb{C}^d) \to \mathrm{L}^2_\sigma(\Omega)$ is compact so that A_2^{-1} is compact as an operator from $\mathrm{L}^2_\sigma(\Omega)$ into $\mathrm{L}^2_\sigma(\Omega)$. Thus, A_2^{-1} has countably many eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n \to 0$ as $n \to \infty$.

Let $\mu \in \mathbb{C} \setminus \{0\}$. Since

$$\mu - A_2^{-1} = (\mu A_2 - \mathrm{Id}) A_2^{-1},$$

we find that $\mu - A_2^{-1}$ is surjective if and only if $1/\mu - A_2$ is surjective. Moreover, since

$$(\mu - A_2^{-1}) u = 0 \qquad \Leftrightarrow \qquad (\mu A_2 - \mathrm{Id}) u = 0,$$

we find that $\mu - A_2^{-1}$ is injective if and only if $1/\mu - A_2$ is injective. It follows that $\rho(A_2) = \{1/\mu : \mu \in \rho(A_2^{-1}) \setminus \{0\}\} \cup \{0\}$, so that $\sigma(A_2) = (\lambda_n)_{n\in\mathbb{N}} := (1/\mu_n)_{n\in\mathbb{N}}$. By Remark 5.2.2 and the invertibility of A_2 all λ_n's are positive real numbers.

Consider the case $p > 2$. As A_p is the part of A_2 in $\mathrm{L}^p_\sigma(\Omega)$, we find that $\lambda - A_p$ is injective if $\lambda - A_2$ is injective. Moreover, if $\lambda - A_2$ is surjective, the embedding $\mathrm{L}^p_\sigma(\Omega) \subset \mathrm{L}^2_\sigma(\Omega)$ implies that for $f \in \mathrm{L}^p_\sigma(\Omega)$ there exists $u \in \mathcal{D}(A_2)$ with

$$(\lambda - A_2)u = f \qquad \Leftrightarrow \qquad A_2 u = \lambda u - f. \tag{5.10}$$

Since $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$, an application of Sobolev's embedding theorem implies that $u \in \mathrm{L}^{2d/(d-2)}(\Omega; \mathbb{C}^d)$. If $p > 2d/(d-2)$, then ε must be that large such that

$$\frac{2d}{d-2} < \frac{2d}{d-1} + \varepsilon. \tag{5.11}$$

By the choice of ε, it follows that the Helmholtz projection is bounded on $\mathrm{L}^{p_1}(\Omega; \mathbb{C}^d)$ with $p_1 := 2d/(d-2)$ by Theorem 5.1.10. By virtue of Lemma 5.2.8, we conclude that $u \in \mathrm{L}^{p_1}_\sigma(\Omega)$, so that $u \in \mathcal{D}(A_{p_1})$ by (5.10). The inclusion $\mathcal{D}(A_{p_1}) \subset \mathrm{W}^{1,p_1}_{0,\sigma}(\Omega)$, follows by the choice of ε and since (5.11) is assumed to hold. Proceed iteratively by Sobolev's embedding theorem and argue as above until the resulting Sobolev index satisfies $p_k \geq p$. Then, we derive that $u \in \mathrm{L}^p_\sigma(\Omega)$ and hence in $\mathcal{D}(A_p)$ and that

$$(\lambda - A_p)u = f.$$

We conclude that $\lambda - A_p$ is surjective if $\lambda - A_2$ is surjective.

Literally the same reasoning as above implies that if $u_n \in \mathcal{D}(A_2)$ is an eigenvector of A_2 with corresponding eigenvalue λ_n, that then $u_n \in \mathcal{D}(A_p)$ is an eigenvector of A_p. We conclude that $\sigma(A_2) = \sigma(A_p)$.

If $p < 2$, we conclude by Schechter [85, Thm. 7.15 & Thm. 7.16] that $\rho(A_{p'}) \subset \rho(A_p)$, which is equivalent to $\sigma(A_p) \subset \sigma(A_{p'}) = \sigma(A_2)$. Moreover, since A_p is the closure of A_2, we have $\mathcal{D}(A_2) \subset \mathcal{D}(A_p)$, so that all eigenvectors of A_2 are eigenvectors of A_p as well. Thus, $\sigma(A_2) \subset \sigma(A_p)$. This concludes the proof for $p < 2$.

Finally, to prove $(\lambda - A_p)^{-1}|_{\mathrm{L}^q_\sigma(\Omega)} = (\lambda - A_q)^{-1}$, we establish that $\mathcal{D}(A_q) \subset \mathcal{D}(A_p)$ and that $A_q u = A_p u$ holds for all $u \in \mathcal{D}(A_q)$. This implies the identity for the resolvents. Assume that $p \geq 2$. Then by virtue of the representations of the domain of the Stokes operator, see Theorems 5.2.3 and 5.2.11, and by Hölder's inequality, we have that $\mathcal{D}(A_q) \subset \mathcal{D}(A_p)$. The distributional representation of A_q and A_p given in these theorems show that $A_q u = A_p u$ for all $u \in \mathcal{D}(A_q)$.

Next, assume that $p < 2$ and $q > 2$. As A_q is the part of A_2 in $\mathrm{L}^q_\sigma(\Omega)$ by definition and A_p is the closure of A_2 in $\mathrm{L}^p_\sigma(\Omega)$ by Proposition 5.2.16, we readily find $\mathcal{D}(A_q) \subset \mathcal{D}(A_2) \subset \mathcal{D}(A_p)$ and $A_q u = A_p u$ for all $u \in \mathcal{D}(A_q)$.

It remains to consider the case $q < 2$. Let $u \in \mathcal{D}(A_q)$. Since A_q is the closure of A_2 in $\mathrm{L}^q_\sigma(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A_2)$ with $u_n \to u$ and $A_2 u_n \to A_q u$ in $\mathrm{L}^q(\Omega;\mathbb{C}^d)$. Hölder's inequality implies that both sequences are Cauchy sequences in $\mathrm{L}^p_\sigma(\Omega)$, so that $u \in \mathcal{D}(A_p)$. Next, there exists a subsequence $(A_2 u_{n_k})_{k\in\mathbb{N}}$, so that $A_2 u_{n_k}$ converges pointwise almost everywhere to $A_q u$. This subsequence has another subsequence $(A_2 u_{n_{k_l}})_{l\in\mathbb{N}}$, which converges pointwise almost everywhere to $A_p u$. We conclude that $A_q u = A_p u$. □

The following theorem are the resolvent estimates established by Shen in [89, Thm. 1.1], which resolve Taylor's conjecture made in [93]. For us, these resolvent estimates are the starting point for the next subsection and for the nonlinear theory.

Theorem 5.2.19 (Shen). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and $\theta \in [0,\pi)$. Then there exists $\varepsilon > 0$ such that for every p satisfying*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

there exists a constant $C > 0$ such that for every $f \in \mathrm{L}^p_\sigma(\Omega)$ and all $\lambda \in \mathrm{S}_\theta$ the inequality

$$|\lambda| \, \|(\lambda + A_p)^{-1} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}$$

holds. In particular, $-A_p$ is the generator of a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$.

5.2.3 Maximal L^q-regularity of the Stokes operator and various L^p-L^q-estimates of the Stokes semigroup

In this subsection, we will prove several consequences of Shen's theorem. First of all, we will review the crucial step Shen performed in order to establish Theorem 5.2.19. An easy consequence of this will be the uniform boundedness of the family $\{|\lambda|^{1/2} (\lambda + A_p)^{-1} \mathbb{P}_p \operatorname{div}\}_{\lambda\in\mathrm{S}_\theta}$ for $p > 2$. This will eventually lead to gradient estimates of the Stokes semigroup for $p < 2$. The next step will be, to use Theorems 5.2.19 and 5.2.6 to derive various types of L^p-L^q-estimates of the Stokes semigroup. Finally, we will modify parts of SHEN'S proof in order to deduce the maximal L^q-regularity of the Stokes operator. We start with formulating SHEN'S crucial step.

In order to prove Theorem 5.2.19, SHEN established for every $\lambda \in \mathrm{S}_\theta$, every $x_0 \in \overline{\Omega}$, $0 < r < c \operatorname{diam}(\Omega)$ (where c is a fixed number independent of λ), and every function $u \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^d)$ that is smooth inside $\Omega \cap B(x_0, 3r)$ and solves together with a function ϕ

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = 0 & \text{in } \Omega \cap B(x_0, 3r) \\ \operatorname{div}(u) = 0 & \text{in } \Omega \cap B(x_0, 3r), \end{cases}$$

the weak reverse Hölder estimate

$$\Big(\frac{1}{r^d} \int_{B(x_0, r) \cap \Omega} |u|^{p_\varepsilon} \, \mathrm{d}x \Big)^{\frac{1}{p_\varepsilon}} \leq C \Big(\frac{1}{r^d} \int_{B(x_0, 2r) \cap \Omega} |u|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}},$$

where $p_\varepsilon := 2d/(d-1) + \varepsilon$, see [89, Lem. 6.2].

For example, a function u satisfies these assumptions if $u = (\lambda + A_2)^{-1} \mathbb{P}_2 f$ for some $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ with $f = 0$ in $\Omega \cap B(x_0, 3r)$.

Note that, since the operator div is "local", the function u defined by $(\lambda + A_2)^{-1} \mathbb{P}_2 \operatorname{div}(F)$ with $F \in \mathrm{L}^2(\Omega; \mathbb{C}^{d \times d})$ and $F = 0$ in $\Omega \cap B(x_0, 3r)$ should have the same properties, so that the weak reverse Hölder estimates would imply the uniform boundedness of the family of operators $\{|\lambda|^{1/2} (\lambda + A_2)^{-1} \mathbb{P}_2 \operatorname{div}\}_{\lambda \in \mathrm{S}_\theta}$. This is made precise in the following theorem.

Theorem 5.2.20. *In the setup of Theorem 5.2.19 but with*

$$2 \leq p < \frac{2d}{d-1} + \varepsilon,$$

the family of operators $\{|\lambda|^{1/2} (\lambda + A_2)^{-1} \mathbb{P}_2 \operatorname{div}\}_{\lambda \in \mathrm{S}_\theta}$ *restricts to a bounded family of operators in* $\mathcal{L}(\mathrm{L}^p(\Omega; \mathbb{C}^{d \times d}), \mathrm{L}^p_\sigma(\Omega))$.

If

$$\frac{2d}{d+1} - \varepsilon < p \leq 2,$$

then $\{|\lambda|^{1/2} \nabla (\lambda + A_2)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ *extends to a bounded family of operators in* $\mathcal{L}(\mathrm{L}^p_\sigma(\Omega), \mathrm{L}^p(\Omega; \mathbb{C}^{d^2}))$.

Proof. In Proposition 5.2.5 we have seen that the family of operators $\{|\lambda|^{1/2}\,(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}\}_{\lambda \in \mathrm{S}_\theta}$ (a priori defined on $\mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^{d \times d})$) extends to a bounded family in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d}), \mathrm{L}^2(\Omega; \mathbb{C}^d))$. Moreover, in the same proposition, it was proven that $\{\nabla(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}\}_{\lambda \in \mathrm{S}_\theta}$ is a bounded family in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d}), \mathrm{L}^2(\Omega; \mathbb{C}^{d^2}))$.

Let $u := (\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F)$ for some $F \in \mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$. Let further $(F_n)_{n\in\mathbb{N}} \subset \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^{d\times d})$ be a sequence with $F_n \to F$ in $\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$. The boundedness results mentioned above imply $(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F_n) \to u$, and $\nabla(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F_n) \to \nabla u$ in L^2. By means of these convergences and an integration by parts, we derive that for all $v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$

$$
\begin{aligned}
\lambda \int_\Omega \langle u, v\rangle \, \mathrm{d}x &+ \int_\Omega \langle \nabla u, \nabla v\rangle \, \mathrm{d}x \\
&= \lim_{n\to\infty} \Big\{ \lambda \int_\Omega \langle (\lambda + A)^{-1}\mathbb{P}_2 \operatorname{div}(F_n), v\rangle \, \mathrm{d}x \\
&\qquad + \int_\Omega \langle \nabla(\lambda + A)^{-1}\mathbb{P}_2 \operatorname{div}(F_n), \nabla v\rangle \, \mathrm{d}x \Big\} \\
&= \lim_{n\to\infty} \int_\Omega \langle \operatorname{div}(F_n), v\rangle \, \mathrm{d}x \\
&= - \lim_{n\to\infty} \sum_{i,j=1}^d \int_\Omega (F_n)_{ij} \overline{\partial_i v_j} \, \mathrm{d}x \\
&= - \sum_{i,j=1}^d \int_\Omega F_{ij} \overline{\partial_i v_j} \, \mathrm{d}x.
\end{aligned}
$$

Let $x_0 \in \overline{\Omega}$ and $0 < r < c \operatorname{diam}(\Omega)$, with the number c from [89, Lem. 6.2]. If F vanishes in $\Omega \cap B(x_0, 3r)$, the calculation above shows that

$$
\lambda \int_\Omega \langle u, v\rangle \, \mathrm{d}x + \int_\Omega \langle \nabla u, \nabla v\rangle \, \mathrm{d}x = 0 \qquad (v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega \cap B(x_0, 3r))).
$$

By virtue of Galdi [34, Thm. IV.4.1], inner regularity shows that u is smooth inside $\Omega \cap B(x_0, 3r)$ and that there exists a smooth pressure function ϕ, which solve

$$
\begin{cases}
\lambda u - \Delta u + \nabla \phi = 0 & \text{in } \Omega \cap B(x_0, 3r) \\
\operatorname{div}(u) = 0 & \text{in } \Omega \cap B(x_0, 3r).
\end{cases}
$$

Because $u = (\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}(F)$, Proposition 5.2.5 shows that u lies in $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$. Next, appeal to [89, Lem. 6.2] to conclude that there exists a constant $C > 0$ depending only on d, θ, and the Lipschitz character of Ω such that

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega}|u|^{p_\varepsilon}\,\mathrm{d}x\right)^{\frac{1}{p_\varepsilon}} \leq C\left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega}|u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

holds. Is was already mentioned at the beginning of the proof that $\{|\lambda|^{1/2}(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}\}_{\lambda\in \mathrm{S}_\theta}$ is bounded in $\mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^{d\times d}),\mathrm{L}^2(\Omega;\mathbb{C}^d))$. We conclude by means of Theorem 3.1.2 with $X := \mathbb{C}^{d\times d}$ and $Y := \mathbb{C}^d$, that for each

$$2 < p < \frac{2d}{d-1} + \varepsilon,$$

each operator $|\lambda|^{1/2}(\lambda + A_2)^{-1}\mathbb{P}_2 \operatorname{div}$ restricts to a bounded operator from $\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})$ into $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ and that the resulting family of operators is uniformly bounded.

To prove the second statement of the theorem, proceed via duality. Let p' denote the Hölder conjugate exponent of p. Then, for $f \in \mathrm{C}^\infty_{c,\sigma}(\Omega)$, we have

$$\begin{aligned} &\| \, |\lambda|^{1/2}\,\nabla(\lambda + A_{p'})^{-1}f\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^{d^2})} \\ &\qquad = \sup_{\substack{g\in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^{d^2}) \\ \|g\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})}\leq 1}} \left| \int_\Omega \langle |\lambda|^{1/2}\,\nabla(\lambda + A_{p'})^{-1}f, g\rangle\,\mathrm{d}x\right|. \end{aligned}$$

Note that the gradient of a vector field $u : \Omega \to \mathbb{C}^d$ is assumed to be the vector $(\nabla u_1, \dots, \nabla u_d)$. Integration by parts then delivers

$$= \sup_{\substack{g\in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^{d^2}) \\ \|g\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})}\leq 1}} \left| \sum_{i,j=1}^d \int_\Omega |\lambda|^{1/2}\,[(\lambda + A_{p'})^{-1}f]_j \overline{\partial_i g_{(j-1)d+i}}\,\mathrm{d}x\right|.$$

Note that the term involving the Stokes operator is independent of i. By virtue of (5.8), we rewrite the terms involving the components of g as

$$\sum_{i=1}^d \overline{\partial_i g_{(j-1)d+i}} = \overline{\operatorname{div}(G)_j},$$

with G being the matrix

$$G := \begin{pmatrix} g_1 & \cdots & g_{(d-1)d+1} \\ \vdots & & \vdots \\ g_d & \cdots & g_{d^2} \end{pmatrix}.$$

It follows that

$$\begin{aligned} &\| \, |\lambda|^{1/2} \, \nabla(\lambda + A_{p'})^{-1} f \|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^{d^2})} \\ &\qquad = \sup_{\substack{g \in \mathrm{C}_c^\infty(\Omega;\mathbb{C}^{d^2}) \\ \|g\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq 1}} \Big| \sum_{i,j=1}^d \int_\Omega \langle |\lambda|^{1/2} \, (\lambda + A_{p'})^{-1} f, \overline{\operatorname{div}(G)} \rangle \, \mathrm{d}x \Big|. \end{aligned}$$

By smoothness of f, we have $(\lambda + A_{p'})^{-1} f = (\lambda + A_2)^{-1} f$, so that by self-adjointness of $\mathbb{P}_2$ and A_2 this coincides with

$$= \sup_{\substack{g \in \mathrm{C}_c^\infty(\Omega;\mathbb{C}^{d^2}) \\ \|g\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq 1}} \Big| \sum_{i,j=1}^d \int_\Omega \langle f, |\lambda|^{1/2} \, (\lambda + A_2)^{-1} \mathbb{P}_2 \overline{\operatorname{div}(G)} \rangle \, \mathrm{d}x \Big|.$$

Finally, use Hölder's inequality as well as the first part of the proof to conclude that $\{|\lambda|^{1/2} \, \nabla(\lambda + A_{p'})^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ defines a uniformly bounded family in $\mathcal{L}(\mathrm{L}^{p'}_\sigma(\Omega), \mathrm{L}^{p'}(\Omega;\mathbb{C}^{d^2}))$. $\square$

The following proposition shows, how the gradient estimates for the resolvent obtained in the previous theorem are transferred into gradient estimates of the corresponding semigroup. For another purpose, we state it in an abstract context.

Proposition 5.2.21. *Let U be a closed subspace of $\mathrm{L}^p(\Omega;\mathbb{C}^N)$ for some $1 < p < \infty$, $N \in \mathbb{N}$, and an open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Let $-B$ be the generator of a bounded analytic semigroup on U with $\mathcal{D}(B) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{C}^N)$. If there exists $\theta > \pi/2$ and $C > 0$ such that $\mathrm{S}_\theta \subset \rho(-B)$ and*

$$\| \, |\lambda|^{\frac{1}{2}} \, \nabla(\lambda + B)^{-1} f \|_{\mathrm{L}^p(\Omega;\mathbb{C}^{dN})} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)} \qquad (f \in U, \lambda \in \mathrm{S}_\theta),$$

then there exists a constant $C' > 0$ such that

$$\|\nabla \mathrm{e}^{-tB} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{dN})} \leq C' t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)} \qquad (f \in U, t > 0).$$

Proof. Use Cauchy's integral formula for analytic semigroups, cf. (2.1), to write

$$\mathrm{e}^{-tB}f = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_t} \mathrm{e}^{\lambda t}(\lambda + B)^{-1} f \,\mathrm{d}\lambda,$$

where γ_t is the path parameterizing the boundary of $\mathrm{S}_\varphi \cup B(0, 1/t)$ for some $\varphi \in (\pi/2, \theta)$. In view of the parameterization of the path, the integral on the right-hand side turns into

$$\begin{aligned}\int_{\gamma_t} \mathrm{e}^{\lambda t}(\lambda + B)^{-1} f \,\mathrm{d}\lambda = &- \int_{t^{-1}}^{\infty} \mathrm{e}^{\varphi\mathrm{i}} \mathrm{e}^{r\mathrm{e}^{\varphi\mathrm{i}}t} (r\mathrm{e}^{\varphi\mathrm{i}} + B)^{-1} f \,\mathrm{d}r \\ &+ \int_{t^{-1}}^{\infty} \mathrm{e}^{-\varphi\mathrm{i}} \mathrm{e}^{r\mathrm{e}^{-\varphi\mathrm{i}}t} (r\mathrm{e}^{-\varphi\mathrm{i}} + B)^{-1} f \,\mathrm{d}r \\ &+ \int_{-\varphi}^{\varphi} \mathrm{i}t^{-1} \mathrm{e}^{\vartheta\mathrm{i}} \mathrm{e}^{\mathrm{e}^{\vartheta\mathrm{i}}} (t^{-1}\mathrm{e}^{\vartheta\mathrm{i}} + B)^{-1} f \,\mathrm{d}\vartheta.\end{aligned}$$

Since $\mathcal{D}(B) \subset \mathrm{W}^{1,p}(\Omega; \mathbb{C}^N)$, we can apply the gradient onto each of the integrands. For the first two integrals, we estimate by means of the gradient estimates of the resolvent

$$\begin{aligned}\int_{t^{-1}}^{\infty} &\|\mathrm{e}^{\pm\varphi\mathrm{i}} \mathrm{e}^{r\mathrm{e}^{\pm\varphi\mathrm{i}}t} \nabla (r\mathrm{e}^{\pm\varphi\mathrm{i}} + B)^{-1} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{dN})} \,\mathrm{d}r \\ &\leq C \int_{t^{-1}}^{\infty} \mathrm{e}^{rt\cos(\varphi)} r^{-\frac{1}{2}} \,\mathrm{d}r \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)}.\end{aligned}$$

Substituting $s = rt$ yields

$$= Ct^{-\frac{1}{2}} \int_1^{\infty} \mathrm{e}^{s\cos(\varphi)} s^{-\frac{1}{2}} \,\mathrm{d}s \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)}.$$

Now, the integral converges, as $\cos(\varphi)$ is negative.

The third integral that appears in the parameterization of the Cauchy integral is estimated again by means of the gradient estimates of the resolvent

$$\begin{aligned}\int_{-\varphi}^{\varphi} &\|\mathrm{i}t^{-1} \mathrm{e}^{\vartheta\mathrm{i}} \mathrm{e}^{\mathrm{e}^{\vartheta\mathrm{i}}} \nabla (t^{-1}\mathrm{e}^{\vartheta\mathrm{i}} + B)^{-1} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{dN})} \,\mathrm{d}\vartheta \\ &\leq Ct^{-\frac{1}{2}} \int_{-\varphi}^{\varphi} \mathrm{e}^{\cos(\vartheta)} \,\mathrm{d}\vartheta \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)}.\end{aligned}$$

Thus, by elementary properties of closed linear operators and the Bochner integral, see, e.g., Arendt, Batty, Hieber, and Neubrander [5, Prop. 1.1.7], we find that

$$\nabla \mathrm{e}^{-tB} f = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_t} \mathrm{e}^{t\lambda} \nabla (\lambda + B)^{-1} f \, \mathrm{d}\lambda$$

and by the previous estimates, that the gradient estimates for the semigroup are valid. □

The following theorem is a collection of several L^p-L^q-estimates of the Stokes semigroup. In the first place, it states the consequences of the previous two results. In the second place, it states L^p-L^q-estimates that can be derived by combining Theorems 5.2.19 and 5.2.6.

Theorem 5.2.22. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Then, there exists $\varepsilon > 0$ such that the following statements are valid.*

(1) For all

$$\frac{2d}{d+1} - \varepsilon < p \leq q < \frac{2d}{d-1} + \varepsilon$$

it holds $\mathrm{e}^{-tA_p}|_{\mathrm{L}^q_\sigma(\Omega)} = \mathrm{e}^{-tA_q}$ and there exists a constant $C > 0$ such that for all $f \in \mathrm{L}^p_\sigma(\Omega)$

$$\|\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \leq C t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (t > 0).$$

(2) For all

$$\frac{2d}{d+1} - \varepsilon < p \leq q < \frac{2d}{d-1} \quad \textit{with} \quad p \leq 2$$

there exists a constant $C > 0$ such that for all $f \in \mathrm{L}^p_\sigma(\Omega)$

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^{d^2})} \leq C t^{-\frac{1}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (t > 0).$$

(3) For all

$$\frac{2d}{d+1} < p \leq q < \frac{2d}{d-1} + \varepsilon \quad \textit{with} \quad q \geq 2$$

there exists $C > 0$ such that for all $F \in \mathrm{L}^p(\Omega; \mathbb{C}^{d \times d})$

$$\|\mathrm{e}^{-tA_p} \mathbb{P}_p \operatorname{div}(F)\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \leq C t^{-\frac{1}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|F\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})} \qquad (t > 0).$$

Proof. For the whole proof, p' and q' denote the Hölder conjugate exponents od p and q.

We start with the consistency of the semigroups on (1). Recall that by (2.1), the semigroup can be represented via the Cauchy integral

$$\mathrm{e}^{-tA_p} f = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\lambda t} (\lambda + A_p)^{-1} f \, \mathrm{d}\lambda.$$

Since $(\lambda + A_p)^{-1}|_{\mathrm{L}^q_\sigma(\Omega)} = (\lambda + A_q)^{-1}$ by Proposition 5.2.18, we directly infer $\mathrm{e}^{-tA_p}|_{\mathrm{L}^q_\sigma(\Omega)} = \mathrm{e}^{-tA_q}$.

We proceed by establishing the L^p-L^q-estimates in (1) and start with $p = 2$. Because $(\mathrm{e}^{-tA_2})_{t \geq 0}$ is an analytic semigroup, the image $\mathrm{e}^{-tA_2} f$ lies in $\mathcal{D}(A_2)$. This domain is continuously included into all domains of A_2^s with $0 \leq s < 1$, see Haase [46, Prop. 3.1.1 c)]. Combining this fact with Theorem 5.2.6, we conclude that $\mathcal{D}(A_2)$ continuously embeds into all spaces $\mathrm{H}^{2s,2}(\Omega; \mathbb{C}^d)$ with $s < 3/4$. Assume in the following, that ε is that small, such that $q < 2d/(d-3)$ (if $d = 3$, simply assume $q < \infty$) is valid. In this case, Sobolev's embedding theorem, see Bergh and Löfström [9, Thm. 6.5.1], implies that $\mathrm{H}^{2s,2}(\Omega; \mathbb{C}^d)$ continuously embeds into $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ for some $s \in [0, \frac{3}{4})$. The numbers q and s are related via

$$q = \frac{2d}{d - 4s} = \Big(\frac{1}{2} - \frac{2s}{d} \Big)^{-1} \quad \Leftrightarrow \quad s = \frac{d}{2} \Big(\frac{1}{2} - \frac{1}{q} \Big).$$

Using that $2 \leq q < 2d/(d-3)$, we directly see that s indeed has to satisfy

$$0 \leq s < \frac{d}{2} \Big(\frac{1}{2} - \frac{d-3}{2d} \Big) = \frac{3}{4}.$$

Consequently

$$\| \mathrm{e}^{-tA_2} f \|_{\mathrm{L}^q(\Omega; \mathbb{C}^d)} \leq C \| \mathrm{e}^{-tA_2} f \|_{\mathrm{H}^{2s,2}(\Omega; \mathbb{C}^d)}.$$

Appealing to Theorem 5.2.6 this is controlled by

$$\leq C \| A_2^s \mathrm{e}^{-tA_2} f \|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)}.$$

Using that $(\mathrm{e}^{-tA_2})_{t \geq 0}$ is a bounded analytic semigroup together with the moment inequality [46, Prop. 6.6.4], shows that this is controlled by

$$\leq C t^{-s} \| f \|_2.$$

Note that s is exactly the desired exponent for $p = 2$ and $q \geq 2$. Moreover, if $p = q$, Theorem 5.2.19 shows that $(\mathrm{e}^{-tA_p})_{t\geq 0}$ is a bounded family of operators on $\mathrm{L}^p_\sigma(\Omega)$, so that it remains to investigate the case $2 < p < q$. Taking ε that small, such that the Helmholtz projection is bounded on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ by means of Theorem 5.1.10, it suffices to consider $(\mathrm{e}^{-tA_p}\mathbb{P}_p)_{t\geq 0}$ as a family of bounded operators from $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ into $\mathrm{L}^q(\Omega; \mathbb{C}^d)$. Write

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$$

for some $\theta \in (0, 1)$ and use complex interpolation, cf. Theorem 1.2.4, to conclude

$$\|\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \leq Ct^{-\frac{d}{2}(\frac{1-\theta}{2} - \frac{1-\theta}{q})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega)).$$

Note that $\frac{1-\theta}{2} = \frac{1}{p} - \frac{\theta}{q}$, so that the desired inequality holds true with the right exponents.

If $2d/(d+1)-\varepsilon < p \leq q \leq 2$ conclude via duality, that for all $f \in \mathrm{L}^2_\sigma(\Omega)$

$$\|\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} = \sup_{\substack{g\in \mathrm{C}^\infty_{c,\sigma}(\Omega) \\ \|g\|_{\mathrm{L}^{q'}(\Omega;\mathbb{C}^d)}\leq 1}} \Big| \int_\Omega \langle \mathrm{e}^{-tA_p} f, g\rangle \,\mathrm{d}x\Big|.$$

Since $f \in \mathrm{L}^2_\sigma(\Omega)$, we have $\mathrm{e}^{-tA_p} f = \mathrm{e}^{-tA_2} f$ and since $g \in \mathrm{C}^\infty_{c,\sigma}(\Omega)$, we have $\mathrm{e}^{-tA_2} g = \mathrm{e}^{-tA_{q'}} g$. Moreover, each of the semigroup operators is self-adjoint by the self-adjointness of A_2, see Haase [46, Lem. 2.6.2]. Thus,

$$= \sup_{\substack{g\in \mathrm{C}^\infty_{c,\sigma}(\Omega) \\ \|g\|_{\mathrm{L}^{q'}(\Omega;\mathbb{C}^d)}\leq 1}} \Big| \int_\Omega \langle f, \mathrm{e}^{-tA_{q'}} g\rangle \,\mathrm{d}x\Big|.$$

By means of Hölder's inequality and $2 \leq q' \leq p' < 2d/(d-1) + \varepsilon$, we estimate by virtue of the previous consideration

$$\leq Ct^{-\frac{d}{2}(\frac{1}{q'} - \frac{1}{p'})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}.$$

Since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q}$, we conclude this case by density of $\mathrm{L}^2_\sigma(\Omega)$ in $\mathrm{L}^p_\sigma(\Omega)$.

Finally, if $2d/(d+1) - \varepsilon < p \leq 2 \leq q < 2d/(d-1) + \varepsilon$, proceed for all $f \in \mathrm{L}^p_\sigma(\Omega)$ by means of the semigroup law

$$\begin{aligned}\|\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} &\leq C t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})} \|\mathrm{e}^{-\frac{t}{2}A_p} f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \\ &\leq C t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}.\end{aligned}$$

This proves the first estimate of the proposition.

In order to conclude (2), we start as in (1) and estimate for all $f \in \mathrm{L}^2_\sigma(\Omega)$ (with a different s)

$$\|\nabla \mathrm{e}^{-tA_2} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^{d^2})} \leq C \|\mathrm{e}^{-tA_2}\|_{\mathrm{H}^{2s}(\Omega;\mathbb{C}^d)} \leq C t^{-s} \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}.$$

Now, s is calculated such that $\mathrm{H}^{2s,2}(\Omega;\mathbb{C}^d)$ embeds into $\mathrm{W}^{1,q}(\Omega;\mathbb{C}^d)$, i.e.,

$$q = \frac{2d}{d - 2(2s-1)} = \Big(\frac{1}{2} - \frac{2s}{d} + \frac{1}{d}\Big)^{-1} \quad \Leftrightarrow \quad s = \frac{1}{2} + \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big),$$

which yields the desired exponent if $p = 2$. Note that for $2 \leq q < 2d/(d-1)$, s has to satisfy

$$\frac{1}{2} \leq s < \frac{1}{2} + \frac{d}{2}\Big(\frac{1}{2} - \frac{d-1}{2d}\Big) = \frac{3}{4},$$

which is a valid value for s in view of Theorem 5.2.6.

If $p < 2$ and $q \geq 2$, we conclude by the semigroup law and (1) that for all $f \in \mathrm{L}^p_\sigma(\Omega)$

$$\begin{aligned}\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^{d^2})} &\leq C t^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})} \|\mathrm{e}^{-\frac{t}{2}A_p} f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \\ &\leq C t^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)},\end{aligned}$$

what gives the right exponent.

Finally, consider the case $q < 2$. If $p = q$, combine Theorem 5.2.20 and Proposition 5.2.21 to conclude that for all $f \in \mathrm{L}^p_\sigma(\Omega)$

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq C t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}$$

holds. Thus, assume that $p < q < 2$ and write

$$\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p}$$

for some $\theta \in (0,1)$. Since $\nabla \mathrm{e}^{-tA_p}\mathbb{P}_p$ is bounded from $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ into $\mathrm{L}^2(\Omega;\mathbb{C}^{d^2})$ and from $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ into $\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})$, the complex interpolation theorem, Theorem 1.2.4, implies that for all $f \in \mathrm{L}^p_\sigma(\Omega)$

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^{d^2})} \leq C t^{-\frac{1-\theta}{2} - \frac{d}{2}(\frac{1-\theta}{p} - \frac{1-\theta}{2}) - \frac{\theta}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}$$

holds. The exponent of t is the desired one, as $\frac{1-\theta}{2} = \frac{1}{q} - \frac{\theta}{p}$.

To prove (3), we proceed by duality. Indeed, we find for all $F \in \mathrm{C}_c^\infty(\Omega;\mathbb{C}^{d\times d})$

$$\|\mathrm{e}^{-tA_p}\mathbb{P}_p \operatorname{div}(F)\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} = \sup_{\substack{f \in \mathrm{C}^\infty_{c,\sigma}(\Omega) \\ \|f\|_{\mathrm{L}^{q'}(\Omega;\mathbb{C}^d)} \leq 1}} \Big| \int_\Omega \langle f, \mathrm{e}^{-tA_p}\mathbb{P}_p \operatorname{div}(F)\rangle \, \mathrm{d}x \Big|.$$

By smoothness of F, we have $\mathbb{P}_p \operatorname{div}(F) \in \mathrm{L}^2_\sigma(\Omega)$, so that by the self-adjointness of A_2

$$= \sup_{\substack{f \in \mathrm{C}^\infty_{c,\sigma}(\Omega) \\ \|f\|_{\mathrm{L}^{q'}(\Omega;\mathbb{C}^d)} \leq 1}} \Big| \int_\Omega \langle \mathrm{e}^{-tA_{q'}} f, \operatorname{div}(F)\rangle \, \mathrm{d}x \Big|.$$

An integration by parts followed by Hölder's inequality yield

$$\leq \sup_{\substack{f \in \mathrm{C}^\infty_{c,\sigma}(\Omega) \\ \|f\|_{\mathrm{L}^{q'}(\Omega;\mathbb{C}^d)} \leq 1}} \|\nabla \mathrm{e}^{-tA_{q'}} f\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^{d^2})} \|F\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})}.$$

Finally, note that $2d/(d-1) - \varepsilon < q' \leq 2$, $q' \leq p'$, and $p' < 2d/(d-1)$, so that we can appeal to (2) to conclude

$$\leq C t^{-\frac{1}{2} - \frac{d}{2}(\frac{1}{q'} - \frac{1}{p'})} \|F\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})}.$$

Since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q}$, we conclude the proof by density of $\mathrm{C}_c^\infty(\Omega;\mathbb{C}^{d\times d})$ in $\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})$. $\square$

The rest of this subsection is dedicated to prove maximal L^q-regularity of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$. For this purpose, recall Theorem 2.3.5 and Proposition 2.3.4. By virtue of these two results, A_p has maximal

L^q-regularity for every $1 < q < \infty$ if there exists $\theta \in (\pi/2, \pi)$ such that the family $\mathcal{T}$ defined by

$$\{\lambda_1(\lambda_1 + A_p)^{-1}\mathbb{P}_p, \dots, \lambda_{n_0}(\lambda_{n_0} + A_p)^{-1}\mathbb{P}_p, 0, \dots) : n_0 \in \mathbb{N}, (\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta\}$$

is bounded in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$. Note that $f \in \mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d))$ can be identified with a sequence $(f_n)_{n\in\mathbb{N}}$ having each component $f_n \in \mathrm{L}^p(\Omega; \mathbb{C}^d)$. In Subsection 2.3, it was defined that an operator $T \in \mathcal{T}$ acts on f via

$$Tf = (\lambda_1(\lambda_1 + A_p)^{-1}\mathbb{P}_p f_1, \dots, \lambda_{n_0}(\lambda_{n_0} + A_p)^{-1}\mathbb{P}_p f_{n_0}, 0, \dots).$$

If $p = 2$, we already know that $\mathcal{T}$ is bounded in $\mathcal{L}(\mathrm{L}^2(\Omega; \ell^2(\mathbb{C}^d)))$. This follows since this boundedness is equivalent to the $\mathcal{R}$-boundedness of the family $\{\lambda(\lambda + A_2)^{-1}\mathbb{P}_2\}_{\lambda\in\mathrm{S}_\theta}$ in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^d))$ by Proposition 2.3.4, which itself is equivalent to the boundedness of this family in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^d))$ by Remark 2.3.2 (1). Note that the latter is a consequence of Proposition 5.2.5.

In order to prove that $\mathcal{T}$ is bounded in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$, we aim to appeal for $p > 2$ and each $T \in \mathcal{T}$ to Theorem 3.1.2. Note that this theorem quantifies the operator norm of T in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$ by means of the constants appearing in the theorem. It was already mentioned in Remark 3.1.3 that if the assumptions of Theorem 3.1.2 can be verified uniformly for each $T \in \mathcal{T}$, i.e., the constants appearing in Theorem 3.1.2 should be the same for each $T \in \mathcal{T}$, then we can bound the operator norm of T in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$ by a uniform constant for all $T \in \mathcal{T}$. We would like to emphasize, that Theorem 3.1.2 is not needed in order to derive the $\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d))$-boundedness of T from its $\mathrm{L}^2(\Omega; \ell^2(\mathbb{C}^d))$-boundedness, because the L^p-boundedness is already known due to the L^p-resolvent estimates in Theorem 5.2.19. However, we need Theorem 3.1.2 in order to calculate the operator norm of each $T \in \mathcal{T}$ in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$ to deduce the boundedness of the whole family $\mathcal{T}$ in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$. Note that we will argue by duality for the case $p < 2$. This is ensured by Remark 2.3.2 (3).

To invoke Theorem 3.1.2, we need the validity of certain weak reverse Hölder estimates. More precisely, if we define $u_n := (\lambda_n + A_2)^{-1}\mathbb{P}_2 f_n$, then the required weak reverse Hölder estimates read as

$$\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,r)} \|Tf\|_{\ell^2(\mathbb{C}^d)}^p \,\mathrm{d}x\bigg)^{\frac{1}{p}} \leq C\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha_1 r)} \|Tf\|_{\ell^2(\mathbb{C}^d)}^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}.$$

Inserting $Tf = (\lambda_1 u_1, \dots, \lambda_{n_0} u_{n_0}, 0, \dots)$ and the definition of the $\ell^2(\mathbb{C}^d)$-norm, reveals that the weak reverse Hölder estimate above looks like

$$\begin{aligned}\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,r)} \Big[\sum_{n=1}^{n_0}(|\lambda_n|\,|u_n|)^2\Big]^{\frac{p}{2}} \,\mathrm{d}x\bigg)^{\frac{1}{p}} \\ \leq C\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha_1 r)} \sum_{n=1}^{n_0}(|\lambda_n|\,|u_n|)^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}.\end{aligned}$$

To establish this kind of estimate, we adapt SHEN'S proof of the weak reverse Hölder estimates given in [89, Thm. 5.6, Lem. 6.1, Lem. 6.2] to this Banach space valued setting.

Lemma 5.2.23. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain, let $\theta \in [0, \pi)$, $(\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta$, and let $p := 2d/(d-1)$.*

Then, there exist constants $C > 0$, $\alpha_2 > \alpha_1 > 1$, $R_0 > 0$, and $\varepsilon > 0$ such that for all $0 < r \leq R_0$ and all $x_0 \in \overline{\Omega}$, which either lie on $\partial\Omega$ or fulfill $B(x_0, \alpha_2 r) \subset \Omega$, the following holds.

Whenever $(f_n)_{n=1}^{n_0} \subset \mathrm{L}^2(\Omega; \mathbb{C}^d)$ with $f_n = 0$ on $\Omega \cap B(x_0, \alpha_2 r)$ and $u_n := (\lambda_n + A_2)^{-1}\mathbb{P}_2 f_n$ the weak reverse Hölder estimate

$$\begin{aligned}\bigg(\frac{1}{r^d}\int_{\Omega\cap B(x_0,r)} \Big[\sum_{n=1}^{n_0}(|\lambda_n|\,|u_n|)^2\Big]^{\frac{p+\varepsilon}{2}} \,\mathrm{d}x\bigg)^{\frac{1}{p+\varepsilon}} \\ \leq C\bigg(\int_{\Omega\cap B(x_0,\alpha_1 r)} \sum_{n=1}^{n_0}(|\lambda_n|\,|u_n|)^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\end{aligned}$$

is valid. All constants C, R_0, α_2, α_1, and ε, depend at most on d, θ, r_0, and the Lipschitz constant M of Ω.

Proof. Let $\alpha_2 > \alpha_1 > 1$ and $R_0 > 0$ to be chosen. Throughout the proof, we assume that $r > 0$ satisfies also $r \leq R_0$. Moreover, let x_0 be either on $\partial\Omega$ or let x_0 be in the interior of Ω such that $B(x_0, \alpha_2 r) \subset \Omega$. Note that since $u_n \in \mathcal{D}(A_2)$, by Theorem 5.2.3 there exists $\pi_n \in \mathrm{L}^2(\Omega)$ such that for all $v \in \mathrm{C}_c^\infty(\Omega; \mathbb{C}^d)$

$$\lambda_n \int_\Omega \langle u_n, v\rangle \,\mathrm{d}x + \int_\Omega \langle \nabla u_n, \nabla v\rangle \,\mathrm{d}x - \int_\Omega \pi_n \overline{\mathrm{div}(v)} \,\mathrm{d}x = \int_\Omega \langle \mathbb{P}_2 f_n, v\rangle \,\mathrm{d}x.$$

Appealing to the representation of $\mathbb{P}_2$, cf. Lemma 5.1.3, and restricting ourselves to $v \in \mathrm{C}_c^\infty(\Omega \cap B(x_0, \alpha_2 r); \mathbb{C}^d)$ shows that the right-hand side coincides with

$$\int_{\Omega\cap B(x_0,\alpha_2 r)} \langle \nabla(-\Delta_N)^{-1} \operatorname{div}(f_n), v\rangle \, \mathrm{d}x.$$

By integration by parts, we find

$$= -\int_{\Omega\cap B(x_0,\alpha_2 r)} (-\Delta_N)^{-1} \operatorname{div}(f_n) \overline{\operatorname{div}(v)} \, \mathrm{d}x,$$

for any representative of $(-\Delta_N)^{-1} \operatorname{div}(f_n)$. By [34, Thm. IV.4.2], it follows that u_n and $\phi_n := \pi_n - (-\Delta_N)^{-1} \operatorname{div}(f_n)$ are smooth and solve

$$\text{(HSRP)} \qquad \begin{cases} \lambda_n u_n - \Delta u_n + \nabla \phi_n = 0 & \text{in } \Omega \cap B(x_0, \alpha_2 r) \\ \operatorname{div}(u_n) = 0 & \text{in } \Omega \cap B(x_0, \alpha_2 r). \end{cases}$$

Case 1: Assume that $B(x_0, \alpha_2 r) \subset \Omega$.

Here, we use the following inequality, proven by Shen in [89, Rem. 5.7]. If u_n and ϕ_n are smooth and solve $\lambda_n u_n - \Delta u_n + \nabla \phi_n = 0$ and $\operatorname{div}(u_n) = 0$ in $B(y,t)$, then there exists a constant $C > 0$ depending only on d and θ such that

$$|\lambda_n|\,|u_n(y)| \leq C\bigg(\frac{1}{t^d}\int_{B(y,t)} (|\lambda_n|\,|u_n|)^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}}. \tag{5.12}$$

We use this estimate for any $y \in B(x_0, r)$ and $t := r$. To do so, we have to assume that $\alpha_2 \geq 2$, because then $B(y,r) \subset B(x_0, \alpha_2 r)$. It follows that

$$\bigg[\sum_{n=1}^{n_0} (|\lambda_n|\,|u_n(y)|)^2\bigg]^{\frac{1}{2}} \leq C\bigg(\sum_{n=1}^{n_0} \frac{1}{r^d}\int_{B(y,r)} (|\lambda_n|\,|u_n|)^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}}.$$

Use $B(y,r) \subset B(x_0, 2r)$ to conclude

$$\leq C\bigg(\frac{1}{r^d}\int_{B(x_0,2r)} \sum_{n=1}^{n_0} (|\lambda_n|\,|u_n|)^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}}.$$

Finally, we take the pth power of this inequality and integrate over $y \in B(x_0, r)$ to get

$$\int_{B(x_0,r)} \Big[\sum_{n=1}^{n_0} (|\lambda_n|\,|u_n(y)|)^2\Big]^{\frac{p}{2}} \,\mathrm{d}y$$
$$\leq C^p\,|B(0,1)|\,r^d \bigg(\frac{1}{r^d}\int_{B(x_0,2r)} \sum_{n=1}^{n_0} (|\lambda_n|\,|u_n|)^2 \,\mathrm{d}x\bigg)^{\frac{p}{2}}.$$

We conclude this case by dividing by r^d and by taking the pth root.
Case 2: Assume that $x \in \partial\Omega$.

In this case, we proceed as in the proof of Proposition 5.1.9. Instead of the intersection of Ω with a ball, we consider the sets $\Omega \cap U_{x_0,r}$ with $U_{x_0,r}$ being a rotation and a translation of the cylinder

$$D(r) = \{(x', x_d) : |x'| < r, |x_d| < 10d(M+1)\}.$$

See the line before (1.5) for the definition of $U_{x_0,r}$. By the definition of $D(r)$, we find that

$$U_{x_0,2r} \subset B(x_0, \tfrac{\alpha_1}{2} r) \tag{5.13}$$

with

$$\alpha_1 := 2\Big[4 + (20d(M+1))^2\Big]^{1/2}.$$

For the rest of this proof, fix the choice $\alpha_2 := 2\alpha_1$. Moreover, we fix the choice of $R_0 := r_0/4$, where r_0 corresponds to Ω via Definition 1.3.1.

Since u_n and ϕ_n solve (HSRP), we find by rescaling that $u_{n,r}(y) := u_n(ry)$ and $\phi_{n,r}(y) := r\phi_n(ry)$ solve the rescaled equations

$$\begin{cases} r^2\lambda_n u_{n,r} - \Delta u_{n,r} + \nabla\phi_{n,r} = 0 & r^{-1}\Omega \cap B(r^{-1}x_0, \alpha_2) \\ \operatorname{div}(u_{n,r}) = 0 & r^{-1}\Omega \cap B(r^{-1}x_0, \alpha_2). \end{cases}$$

Fix $s \in [1,2)$. As it was described in the proof of Proposition 5.1.9, the rescaled set $r^{-1}[\Omega \cap U_{x_0,sr}]$ is given by

$$\{r^{-1}x_0\} + r^{-1}R_{x_0}^{-1} D_{r^{-1}\eta_{x_0}(r\cdot)}(s) =: U^{\mathrm{resc},+}_{x_0,s}.$$

Moreover, it was noted that $r^{-1}\eta_{x_0}(r\cdot)$ has the same Lipschitz constant as η_{x_0}. Appealing to Lemma 1.3.25, we find that the Lipschitz character of the sets $U_{x_0,s}^{\mathrm{resc},+}$ is the same for all $s \in [1,2)$ and all $x_0 \in r^{-1}\partial\Omega$.

We collect some properties of $u_{n,r}$. Since $u_n \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$, it is clear that $u_{n,r} \in \mathrm{W}^{1,2}_{0,\sigma}(r^{-1}\Omega)$. Thus, we can define $g_n^s \in \mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+}; \mathbb{C}^d)$ as the trace of $u_{n,r}$ on $\partial U_{x_0,s}^{\mathrm{resc},+}$. Note that g_n^s vanishes on $r^{-1}\partial\Omega \cap \partial U_{x_0,s}^{\mathrm{resc},+}$. Moreover, by the divergence theorem, see Ziemer [102, Thm. 5.8.2, Rem. 5.8.3], we find that

$$0 = \int_{U_{x_0,s}^{\mathrm{resc},+}} \operatorname{div}(u_{n,r}) \, \mathrm{d}y = \int_{\partial U_{x_0,s}^{\mathrm{resc},+}} \langle \nu_{U_{x_0,s}^{\mathrm{resc},+}}, g_n^s \rangle \, \mathrm{d}\sigma_{U_{x_0,s}^{\mathrm{resc},+}},$$

where $\nu_{U_{x_0,s}^{\mathrm{resc},+}}$ and $\sigma_{U_{x_0,s}^{\mathrm{resc},+}}$ denote the outward unit normal and the surface measure on $\partial U_{x_0,s}^{\mathrm{resc},+}$. Recalling the definition of L^2_ν in (4.1), it follows that $g_n^s \in \mathrm{L}^2_\nu(\partial U_{x_0,s}^{\mathrm{resc},+})$.

Let v_n^s be the solution of the L^2-Dirichlet probem of the Stokes resolvent

$$\begin{cases} r^2 \lambda_n v_n^s - \Delta v_n^s + \nabla \phi_n^s = 0 & \text{in } U_{x_0,s}^{\mathrm{resc},+} \\ \operatorname{div}(v_n^s) = 0 & \text{in } U_{x_0,s}^{\mathrm{resc},+} \\ v_n^s = g_n^s & \text{non-tangentially on } \partial U_{x_0,s}^{\mathrm{resc},+} \\ N_a v_n^s \in \mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+}), & \end{cases}$$

provided by Theorem 4.1.7. Proceed as in the first lines of the proof of [89, Thm. 5.6] to deduce that $u_{n,r} = v_n^s$ on $U_{x_0,s}^{\mathrm{resc},+}$. Consequently, the estimate given in Theorem 4.1.7 is available for $u_{n,r}$. Then, since $U_{x_0,1}^{\mathrm{resc},+} \subset U_{x_0,s}^{\mathrm{resc},+}$

$$\int_1^2 \bigg(\int_{U_{x_0,1}^{\mathrm{resc},+}} \Big[\sum_{n=1}^{n_0} (|\lambda_n| \, |u_{n,r}|)^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \bigg)^{\frac{2}{p}} \, \mathrm{d}s \\ \leq \int_1^2 \bigg(\int_{U_{x_0,s}^{\mathrm{resc},+}} \Big[\sum_{n=1}^{n_0} (|\lambda_n| \, |u_{n,r}|)^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \bigg)^{\frac{2}{p}} \, \mathrm{d}s.$$

Invoke Lemma 5.1.6 to obtain

$$\leq C \int_1^2 \int_{\partial U_{x_0,s}^{\mathrm{resc},+}} \bigg[N_1 \bigg(\Big[\sum_{n=1}^{n_0} (|\lambda_n| \, |u_{n,r}|)^2 \Big]^{\frac{1}{2}} \bigg) \bigg]^2 \, \mathrm{d}\sigma \, \mathrm{d}s.$$

The following few lines are the key step in the adaption of Shen's proof to the Banach space valued setting.

By the definition of N_1, cf. (4.13), we find for all $q \in \partial U^{\text{resc},+}_{x_0,s}$

$$\Big[N_1\Big(\Big[\sum_{n=1}^{n_0}(|\lambda_n|\,|u_{n,r}|)^2\Big]^{\frac{1}{2}}\Big)(q)\Big]^2 = \Big[\sup_{x\in\gamma_1(q)}\Big[\sum_{n=1}^{n_0}(|\lambda_n|\,|u_{n,r}(x)|)^2\Big]^{\frac{1}{2}}\Big]^2.$$

Since the supremum is taken over a non-negative function and since the function $y \mapsto y^2$ is continuous and monotonely increasing on the non-negative real axis, we can interchange the square and the supremum, so that

$$= \sup_{x\in\gamma_1(q)}\sum_{n=1}^{n_0}(|\lambda_n|\,|u_{n,r}(x)|)^2.$$

Using that the supremum over a sum of functions is less than the sums of the suprema of the functions and interchanging the square and the supremum again, yields by definition of N_1

$$\leq \sum_{n=1}^{n_0}|\lambda_n|^2\Big[N_1u_{n,r}(q)\Big]^2.$$

Using this in the estimation above, delivers

$$\int_1^2\Big(\int_{U^{\text{resc},+}_{x_0,1}}\Big[\sum_{n=1}^{n_0}(|\lambda_n|\,|u_{n,r}|)^2\Big]^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{2}{p}}\,\mathrm{d}s$$
$$\leq C\sum_{n=1}^{n_0}|\lambda_n|^2\int_1^2\int_{\partial U^{\text{resc},+}_{x_0,s}}\Big[N_1u_{n,r}\Big]^2\,\mathrm{d}\sigma_{U^{\text{resc},+}_{x_0,s}}\,\mathrm{d}s.$$

Next, a combination of Proposition 4.1.11 with the estimate in Theorem 4.1.7 gives another constant C, depending only on d, θ, and the Lipschitz character of $U^{\text{resc},+}_{x_0,s}$ (which is the same for all $U^{\text{resc},+}_{x_0,t}$, $t \in [1,2)$, due to Lemma 1.3.25) such that

$$\leq C\sum_{n=1}^{n_0}|\lambda_n|^2\int_1^2\int_{\partial U^{\text{resc},+}_{x_0,s}}|g_n^s|^2\,\mathrm{d}\sigma_{U^{\text{resc},+}_{x_0,s}}\,\mathrm{d}s.$$

Recall that g_n^s is the trace of $u_{n,r}$ on $\partial U_{x_0,s}^{\mathrm{resc},+}$ and that g_n^s vanishes on $r^{-1}\partial\Omega \cap \partial U_{x_0,s}^{\mathrm{resc},+}$, as this is the case for $u_{n,r}$. Moreover, since $u_{n,r}$ is smooth inside $U_{x_0,2}^{\mathrm{resc},+}$, we can replace g_n^s by the pointwise evaluation of $u_{n,r}$ on $\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega$. Altogether,

$$= C \sum_{n=1}^{n_0} |\lambda_n|^2 \int_1^2 \int_{\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega} |u_{n,r}|^2 \, \mathrm{d}\sigma_{U_{x_0,s}^{\mathrm{resc},+}} \, \mathrm{d}s.$$

Finally, the same integration argument (involving the co-area formula) as in Proposition 5.1.9 shows that this can be estimated by

$$\leq C \int_{U_{x_0,2}^{\mathrm{resc},+}} \sum_{n=1}^{n_0} (|\lambda_n| \, |u_{n,r}|)^2 \, \mathrm{d}x.$$

Note that on the left-hand side of this inequality, the integration over s simply contributes a factor 1. Thus, taking the square root of this inequality and performing the linear transformation $x = ry$, delivers

$$\begin{aligned} &\Big(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} \Big[\sum_{n=1}^{n_0} (|\lambda_n| \, |u_n|)^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{1}{p}} \\ &\qquad\qquad \leq C \Big(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,2r}} \sum_{n=1}^{n_0} (|\lambda_n| \, |u_n|)^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}. \end{aligned}$$

Since $B(x_0, r) \subset U_{x_0,r}$, the left-hand side is estimated from below by

$$\Big(\frac{1}{r^d} \int_{\Omega \cap B(x_0,r)} \Big[\sum_{n=1}^{n_0} (|\lambda_n| \, |u_n|)^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{1}{p}}$$

and by (5.13), the right-hand side is bounded from above by

$$C \Big(\frac{1}{r^d} \int_{\Omega \cap B(x_0, \frac{\alpha_1}{2} r)} \sum_{n=1}^{n_0} (|\lambda_n| \, |u_n|)^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Having a look onto the statement of the lemma, we would like to replace p by $p + \varepsilon$ on the left-hand side of the weak reverse Hölder estimate. This follows by the self-improving property of weak reverse Hölder estimates, see Proposition 3.1.4. The verification of the assumptions of this proposition is performed exactly as in "Step 1: Getting an ε more" in the proof of Theorem 5.1.10, so that we skip it at this occasion. $\square$

Now, we are ready to combine the previous lemma with the discussion before this lemma, to conclude the maximal L^q-regularity of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$.

Theorem 5.2.24. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and $1 < q < \infty$. Then, there exists $\varepsilon > 0$ such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the Stokes operator A_p has maximal L^q-regularity.

Proof. By virtue of Proposition 5.2.5 as well as the boundedness of the Helmholtz projection on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$, we know that for each $\theta \in [0, \pi)$ the family of operators $\{\lambda(\lambda + A_2)^{-1}\mathbb{P}_2\}_{\lambda \in \mathrm{S}_\theta}$ is bounded. Moreover, by Remark 2.3.2 (1), we conclude that this family is also $\mathcal{R}$-bounded. By Proposition 2.3.4 this is equivalent to the fact, that the family $\mathcal{T}$ of operators

$$\{(\lambda_1(\lambda_1 + A_2)^{-1}\mathbb{P}_2, \dots, \lambda_{n_0}(\lambda_{n_0} + A_2)^{-1}\mathbb{P}_2, 0, \dots) : n_0 \in \mathbb{N}, (\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta\}$$

is bounded in $\mathcal{L}(\mathrm{L}^2(\Omega; \ell^2(\mathbb{C}^d)))$. Let $T \in \mathcal{T}$. Then for $x \in \Omega$ and $f = (f_n)_{n=1}^\infty \in \mathrm{L}^2(\Omega; \ell^2(\mathbb{C}^N))$

$$\|[Tf](x)\|_{\ell^2(\mathbb{C}^d)} = \Big[\sum_{n=1}^{n_0} \Big| \lambda_n [(\lambda_n + A_2)^{-1}\mathbb{P}_2 f_n](x) \Big|^2 \Big]^{\frac{1}{2}}.$$

Thus, by Lemma 5.2.23, T satisfies weak reverse Hölder estimates with implicit constants, that are uniform for all operators in $\mathcal{T}$. By Theorem 3.1.2, it follows that T restricts to a bounded operator on $\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d))$ for every $2 \leq p < 2d/(d-1) + \varepsilon$. Here, ε is the number from Lemma 5.2.23. Appealing to Remark 3.1.3, it follows that $\mathcal{T}$ restricts to a bounded family in $\mathcal{L}(\mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^d)))$. By virtue of Proposition 2.3.4, this is equivalent to the $\mathcal{R}$-boundedness of the family $\{\lambda(\lambda + A_p)^{-1}\mathbb{P}_p\}_{\lambda \in \mathrm{S}_\theta}$ in $\mathcal{L}(\mathrm{L}^p(\Omega; \mathbb{C}^d))$. Since $\mathbb{P}_p$ is the identity on L^p_σ the $\mathcal{R}$-boundedness of the family $\{\lambda(\lambda + A_p)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ in $\mathcal{L}(\mathrm{L}^p_\sigma(\Omega))$ is evident.

Finally, since $1 < p < \infty$, Remark 2.3.2 (3) shows that $\mathcal{R}$-boundedness dualizes. Since for $2d/(d+1) - \varepsilon < p \leq 2$ the antidual space of $\mathrm{L}^{p'}_\sigma(\Omega)$ is isomorphic to $\mathrm{L}^p_\sigma(\Omega)$ with respect to an isomorphism Φ, see Lemma 5.2.13, and since A_p is by definition the operator $\Phi^{-1} A_{p'} \Phi$, we conclude that $\{\lambda(\lambda + A_p)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\mathrm{L}^p_\sigma(\Omega))$.

Since θ can especially be taken to be larger than $\pi/2$, Theorem 2.3.5 implies the maximal L^q-regularity of A_p for every $1 < q < \infty$. $\square$

As we are going to employ the preceding results in the remaining treatise, we will meet the convention that $\varepsilon > 0$ will always be the mimimum of all $\varepsilon > 0$ appeared in this and the latter section. We will refer to this ε by the following convention.

Convention 5.2.25. Let $\varepsilon > 0$ be the minimum of the ε's appearing in the statements of the results of Sections 5.1 and 5.2.

5.3 A discussion on gradient estimates of the Stokes semigroup on L^p_σ

We have already seen in Theorem 5.2.22, that for $2d/(d+1) - \varepsilon < p \leq 2$, the Stokes semigroup satisfies the gradient estimates

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq C t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega), t > 0).$$

This estimate is highly desirable for $p > 2$, as once it is proven in the three dimensional case for $p = 3$, it enables oneself to start Kato's iteration procedure in order to tackle the solvability question of the Navier-Stokes equations in critical spaces, see Kato [59] or Wiegner [100, Sec. 6]. On bounded and smooth domains these estimates were established by Giga and Miyakawa [42, Prop. 1.2, Prop. 1.4]. Unfortunately, in the situation of bounded Lipschitz domains there are some obstacles, that could not be overcome until now and that will be addressed in this section.

5.3.1 Gradient estimates via an embedding of $\mathcal{D}(A_p^{1/2})$ into $\mathrm{W}^{1,p}(\Omega;\mathbb{C}^d)$

We start by sketching the situation for bounded and smooth domains. Here, Giga and Miyakawa proceeded in two steps.

The first step was to use abstract functional analytic results to conclude that one can estimate

$$\|A_p^{\frac{1}{2}} \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega), t > 0).$$

This estimate holds for every bounded analytic semigroup by combining the moment inequality, see Haase [46, Prop. 6.6.4], with the uniform boundedness of $tA_p\mathrm{e}^{-tA_p}$, which characterizes generators of bounded analytic semigroups, see Engel and Nagel [25, Thm. 4.6].

The second step was to show that the continuous embedding

$$\mathcal{D}(A_p^{\frac{1}{2}}) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{C}^d)$$

is valid. This was shown by using that $\mathcal{D}(A_p^{1/2}) = \mathcal{D}((-\Delta_p)^{1/2})^d \cap \mathrm{L}^p_\sigma(\Omega)$, see Giga [40, Thm. 3] and by invoking the respective embedding for the square root of the negative Laplacian, i.e.,

$$\mathcal{D}((-\Delta_p)^{\frac{1}{2}}) \subset \mathrm{W}^{1,p}(\Omega),$$

see Fujiwara [33].

Combining both steps, we find for all $f \in \mathrm{L}^p_\sigma(\Omega)$ and $t > 0$ that

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq C\|A_p^{\frac{1}{2}}\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq Ct^{-\frac{1}{2}}\|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}.$$

Note that $0 \in \rho(A_p)$ is essential, as it allows oneself to endow $\mathcal{D}(A_p)$ with the homogeneous graph norm. The invertibility of A_p for smooth domains is known due to Cattabriga [16].

Let us turn to the situation on bounded Lipschitz domains. Here, Shen proved that $-A_p$ generates a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$ whenever

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon,$$

see Theorem 5.2.19. Thus, the first step of Giga and Miyakawa is already established.

For the second step, the embedding $\mathcal{D}(A_p^{1/2}) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{C}^d)$ is required. To establish this embedding, one could try to imitate the approach of Shen, see [87, Lem. 3.5], to show that the domain of the square root of the negative Dirichlet Laplacian continuously embeds into $\mathrm{W}^{1,p}(\Omega)$. This was proven by Shen for $p > 2$ in the range above. In this case, he argued via duality and performed the following calculation. With p' being the Hölder conjugate exponent of p, he begins with

$$\|(-\Delta_{p'})^{-\frac{1}{2}}\operatorname{div}(f)\|_{\mathrm{L}^{p'}(\Omega)} = \|(-\Delta_{p'})^{\frac{1}{2}}(-\Delta_{p'})^{-1}\operatorname{div}(f)\|_{\mathrm{L}^{p'}(\Omega)}.$$

Then, he uses the continuous embedding $\mathrm{W}^{1,p'}_0(\Omega) \subset \mathcal{D}((-\Delta_{p'})^{\frac{1}{2}})$ to estimate

$$\leq C \|\nabla(-\Delta_{p'})^{-1} \operatorname{div}(f)\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)}.$$

Finally, he uses the boundedness of $\nabla(-\Delta_{p'})^{-1} \operatorname{div}$ on $\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)$ to get

$$\leq C\|f\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)}.$$

Dualizing this estimate yields

$$\|\nabla(-\Delta_p)^{-\frac{1}{2}} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C\|f\|_{\mathrm{L}^p(\Omega)} \qquad (f \in \mathrm{L}^p(\Omega))$$

and thus that $\mathcal{D}((-\Delta_p)^{1/2})$ continuously embeds into $\mathrm{W}^{1,p}(\Omega)$.

Let us see, what happens if we replace $-\Delta_p$ by the Stokes operator A_p. By Theorem 5.2.9, we find that

$$\nabla A_p^{-1} \mathbb{P}_p \operatorname{div}$$

defines a bounded operator from $\mathrm{L}^p(\Omega;\mathbb{C}^{d\times d})$ into $\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})$. Since $A_{p'}$ is defined via duality, a duality argument shows that

$$\nabla A_{p'}^{-1} \mathbb{P}_{p'} \operatorname{div}$$

is bounded from $\mathrm{L}^{p'}(\Omega;\mathbb{C}^{d\times d})$ into $\mathrm{L}^{p'}(\Omega;\mathbb{C}^{d^2})$. Thus, the last estimate of Shen's derivation is no problem for the Stokes operator, so that it remains to prove the continuous embedding

$$\mathrm{W}^{1,p'}_{0,\sigma}(\Omega;\mathbb{C}^d) \subset \mathcal{D}(A_{p'}^{\frac{1}{2}}). \tag{5.14}$$

We would like to get a feeling of what one has to show in order to establish (5.14).

Recall that $\mathcal{D}(A_2)$ is given by

$$\{u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \exists \pi \in \mathrm{L}^2(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^2_\sigma(\Omega)\}$$

by Theorem 5.2.3 with $A_2 u = -\Delta u + \nabla\pi$. Moreover, by Corollary 5.2.12, we know that $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ is contained in $\mathcal{D}(A_2)$ and that

$$A_2 v = -\mathbb{P}_2 \Delta v \qquad (v \in \mathrm{C}^\infty_{c,\sigma}(\Omega)). \tag{5.15}$$

Next, for $u \in \mathrm{W}^{2,p'}_{0,\sigma}(\Omega)$ (the closure of $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ in the $\mathrm{W}^{2,p'}$-norm), let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty_{c,\sigma}(\Omega)$ be an appropriate sequence that approximates u in $\mathrm{W}^{2,p'}(\Omega;\mathbb{C}^d)$. By (5.15) and the fact that $\mathbb{P}_2$ extends to a bounded operator on $\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)$ by Theorem 5.1.10, we find that $(A_2 u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^{p'}_\sigma(\Omega)$. Consequently, $u \in \mathcal{D}(A_{p'})$ since $A_{p'}$ is the closure of A_2 in $\mathrm{L}^{p'}_\sigma(\Omega)$ by Proposition 5.2.16. Additionally, we find that

$$A_{p'} u = -\mathbb{P}_{p'} \Delta u$$

and hence, that

$$\|A_{p'} u\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^d)} \leq C \|u\|_{\mathrm{W}^{2,p'}(\Omega;\mathbb{C}^d)}.$$

The continuous embedding

$$\mathrm{W}^{2,p'}_{0,\sigma}(\Omega) \subset \mathcal{D}(A_{p'})$$

follows. We see, that a similar embedding as (5.14), but for the full operator and not its square root holds true. In the following, we show how to obtain (5.14) under a certain assumption on $A_{p'}$.

It was proven by Mitrea and Monniaux [75, Prop. 2.10 & Thm. 2.12] that

$$[\mathrm{L}^{p'}_\sigma(\Omega), \mathrm{W}^{2,p'}_{0,\sigma}(\Omega)]_{1/2} = \mathrm{W}^{1,p'}_{0,\sigma}(\Omega).$$

Next, assume that $(1 + A_{p'})^{\mathrm{i}s}$ is bounded for all $s \in \mathbb{R}$. In this case, the domain of the square root of $A_{p'}$ can be expressed via complex interpolation as

$$\mathcal{D}(A_{p'}^{\frac{1}{2}}) = [\mathcal{D}(A_{p'}^0), \mathcal{D}(A_{p'})]_{1/2},$$

see Haase [46, Thm. 6.6.9]. Combining this with the previous interpolation identity and the continuous embedding of $\mathrm{W}^{2,p'}_{0,\sigma}(\Omega)$ into $\mathcal{D}(A_{p'})$ delivers the desired embedding

$$\mathrm{W}^{1,p'}_{0,\sigma}(\Omega) = [\mathrm{L}^{p'}_\sigma(\Omega), \mathrm{W}^{2,p'}_{0,\sigma}(\Omega)]_{1/2} \subset [\mathcal{D}(A_{p'}^0), \mathcal{D}(A_{p'})]_{1/2} = \mathcal{D}(A_{p'}^{\frac{1}{2}}).$$

Note that in the literature one says that $1 + A_{p'}$ has *bounded imaginary powers* if $(1+A_{p'})^{\mathrm{i}s}$ is bounded for all $s \in \mathbb{R}$. A sufficient condition for the fact that $1 + A_{p'}$ has bounded imaginary powers is, that the H^∞-calculus of $A_{p'}$ or $1 + A_{p'}$ is bounded. Thus, we can record the following theorem.

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and let*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

with $\varepsilon > 0$ from Convention 5.2.25. If $1 + A_{p'}$ has bounded imaginary powers, or, if the H^∞-calculus of $A_{p'}$ is bounded, then the Stokes semigroup satisfies the gradient estimates

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq C t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega), t > 0).$$

5.3.2 Gradient estimates via gradient estimates of the resolvent

In Proposition 5.2.21, we have seen that the estimates

$$\||\lambda|^{\frac{1}{2}} (\lambda + A_p)^{-1} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{d^2})} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega), \lambda \in \mathrm{S}_\theta)$$

for some $\theta \in (\pi/2, \pi)$, lead to the gradient estimates of the corresponding semigroup. In the case $p < 2$, these gradient estimates for the Stokes resolvent were established in Theorem 5.2.20. To get a feeling for what one could do, in order to establish these estimates for $p > 2$, we will scetch the elliptic situation in the following. Afterwards, we will try to imitate this "blueprint" for the Stokes operator and point out the difficulties, that arise.

Note that in [88], Shen proved for second order elliptic systems $B = -\operatorname{div} \mu \nabla$, with constant and symmetric coefficients, that $\sigma(B) \subset [0, \infty)$, and that for every $\theta \in [0, \pi)$ and $\lambda \in \mathrm{S}_\theta$ the estimate

$$\| \, |\lambda|^{\frac{1}{2}} \, \nabla (\lambda + B)^{-1} f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^{dN})} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^N)} \qquad (f \in \mathrm{L}^p(\Omega;\mathbb{C}^N)).$$

holds. Here, N denotes the number of equations in the system $-\operatorname{div} \mu \nabla$ and p satisfies

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

for some $\varepsilon > 0$ depending on d, θ, and some geometric quantities. Note that Shen considered Dirichlet boundary conditions; the same was proven for Neumann boundary conditions by Wei and Zhang in [97].

Note that the "blueprint" presented here does not follow the lines of Shen performed in [88], but that it uses more modern techniques developed by Shen since then. Further, note that we do not go into every detail, as we only intend to present the major steps.

Let Ω be a bounded Lipschitz domain and let $B = -\operatorname{div}\mu\nabla$ be an elliptic system on Ω with constant and symmetric coefficients complemented with Dirichlet boundary conditions. The L^2-theory of this operator is straightforward. For example, it follows literally as in Proposition 5.2.5 that $\sigma(B) \subset (0,\infty)$ and that for each $\theta \in [0,\pi)$ the family of operators

$$\{|\lambda|^{\frac{1}{2}} \nabla(\lambda + B)^{-1}\}_{\lambda \in \mathrm{S}_\theta} \subset \mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^N), \mathrm{L}^2(\Omega;\mathbb{C}^{dN}))$$

is bounded. Now, it is the goal to verify the assumptions of the L^p-extrapolation theorem, Theorem 3.1.2, with $X = \mathbb{C}^N$ and $Y = \mathbb{C}^{dN}$ uniformly in $\lambda \in \mathrm{S}_\theta$. Note that in order to establish the weak reverse Hölder estimates, we have already seen in Sections 5.1 and 5.2 that it is possible to replace balls centered on $\partial\Omega$ by the sets $\Omega \cap U_{x_0,r}$.

For this purpose, let $0 < r \leq r_0/5$ and $x_0 \in \partial\Omega$, r_0 belonging to Ω via Definition 1.3.1. We proceed with the following steps.

(1) Let $\lambda \in \mathrm{S}_\theta$ and $f \in \mathrm{L}^2(\Omega;\mathbb{C}^N)$ with $f = 0$ on $\Omega \cap U_{x_0,\alpha_2 r}$ for some α_2 large enough. Define $u := (\lambda + B)^{-1} f$ and note that $u \in \mathrm{W}^{1,2}(U_{x_0,3r};\mathbb{C}^N) \cap \mathrm{W}^{1,2}_0(\Omega;\mathbb{C}^N)$. Rescale the whole situation to find that

$$\begin{aligned} && \lambda u - \operatorname{div}\mu\nabla u = 0 \qquad & \text{in } \Omega \cap U_{x_0,3r} \\ \Leftrightarrow && r^2\lambda u_r - \operatorname{div}\mu\nabla u_r = 0 \qquad & \text{in } U^{\mathrm{resc},+}_{x_0,3}, \end{aligned}$$

where $u_r(y) := u(ry)$ and $U^{\mathrm{resc}}_{x_0,s} := r^{-1}[\Omega \cap U_{x_0,sr}]$ is the set

$$U^{\mathrm{resc}+}_{x_0,s} = \{r^{-1}x_0\} + r^{-1}R^{-1}_{x_0} D_{r^{-1}\eta_{x_0}(r\cdot)}(s) \qquad (s \in (0,3]).$$

By inner regularity, we may assume that u_r is smooth inside $U^{\mathrm{resc},+}_{x_0,3}$.

(2) As $u_r \in \mathrm{W}^{1,2}(U^{\mathrm{resc},+}_{x_0,3})$, one can take the trace $g_s \in \mathrm{L}^2(\partial U^{\mathrm{resc},+}_{x_0,s}; \mathbb{C}^N)$ of u_r on $\partial U^{\mathrm{resc},+}_{x_0,s}$ for every $s \in (0,3]$. Note that g_s vanishes on $r^{-1}\partial\Omega \cap \partial U^{\mathrm{resc},+}_{x_0,s}$, as u vanishes on $\partial\Omega$ due to the Dirichlet boundary conditions. Let v_s be the solution of the L^2-Dirichlet problem

$$\begin{cases} r^2\lambda v_s - \operatorname{div} \mu \nabla v_s = 0 & \text{in } U^{\mathrm{resc},+}_{x_0,s} \\ \qquad\qquad\qquad v_s = g_s & \text{non-tangentially on } \partial U^{\mathrm{resc},+}_{x_0,s} \\ \qquad\quad (v_s)^* \in \mathrm{L}^2(\partial U^{\mathrm{resc},+}_{x_0,s}), \end{cases}$$

provided by [88, Thm. 0.10]. One can show that v_s and u_r coincide in $U^{\mathrm{resc},+}_{x_0,s}$ by proceeding literally as in the first lines of the proof of [89, Thm. 5.6] (where Shen proved the corresponding result for the Stokes resolvent problem instead of the elliptic resolvent problem). Thus, the non-tangential behavior is valid for u_r.

(3) By (1) and (2), we have $u_r \in \mathrm{W}^{1,2}(U^{\mathrm{resc},+}_{x_0,3}, \mathbb{C}^N)$, $(u_r)^* \in \mathrm{L}^2(\partial U^{\mathrm{resc},+}_{x_0,3})$, and that u_r converges to zero non-tangentially on $r^{-1}\partial\Omega \cap U^{\mathrm{resc},+}_{x_0,3}$. Thus, by virtue of Proposition 1.3.28, we conclude that there exists a set of measure zero $\mathcal{N} \subset [1,2]$ such that $u_r \in \mathrm{W}^{1,2}(\partial U^{\mathrm{resc},+}_{x_0,s}; \mathbb{C}^N)$ for every $s \in [1,2] \setminus \mathcal{N}$. Moreover, since g_s is the trace of u_r onto $\partial U^{\mathrm{resc},+}_{x_0,s}$ and since u_r is smooth in $U^{\mathrm{resc},+}_{x_0,3}$, we conclude that u_r and g_s coincide almost everywhere on $\partial U^{\mathrm{resc},+}_{x_0,s}$ (with respect to the surface measure).

Thus, by the regularity theory of the L^2-Dirichlet problem of the elliptic system, see Shen [88, Thm. 0.12], there exists a constant $C > 0$, depending only on the ellipticity constant, d, r_0, and the Lipschitz character of $U^{\mathrm{resc},+}_{x_0,s}$, such that

$$\begin{aligned} \|(\nabla u_r)^*\|_{\mathrm{L}^2(\partial U^{\mathrm{resc}}_{x,s} \cap r^{-1}\Omega)} \leq C\Big\{ &\|\nabla_{\tan} u_r\|_{\mathrm{L}^2(\partial U^{\mathrm{resc},+}_{x_0,s} \setminus r^{-1}\partial\Omega; \mathbb{C}^{dN})} \\ &+ (1 + |r^2\lambda|^{\frac{1}{2}}) \|u_r\|_{\mathrm{L}^2(\partial U^{\mathrm{resc},+}_{x_0,s} \setminus r^{-1}\partial\Omega; \mathbb{C}^N)} \Big\}. \end{aligned}$$

Note that by Lemma 1.3.25 the Lipschitz character of $U^{\mathrm{resc},+}_{x_0,s}$ is comparable to d and M, since $s \in [1,2]$.

(4) Performing the same integration argument presented in the proofs

of Proposition 5.1.9 and Lemma 5.2.23 leads to the estimate

$$\begin{aligned}\Big(\int_{U_{x_0,1}^{\mathrm{resc},+}} |\nabla u_r|^{\frac{2d}{d-1}} \,\mathrm{d}y \Big)^{\frac{d-1}{2d}} \\ \leq C \bigg(\int_{U_{x_0,2}^{\mathrm{resc},+}} |\nabla u_r|^2 + \Big(1 + |r^2 \lambda|^{\frac{1}{2}}\Big)^2 |u_r|^2 \,\mathrm{d}y \bigg)^{\frac{1}{2}}.\end{aligned}$$

Next, appeal to the inequality $(1 + |r^2\lambda|^{\frac{1}{2}})^2 \leq 2 + 2|r^2\lambda|$ and then and use the linear transformation $x = ry$ to obtain

$$\begin{aligned}\Big(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\nabla u|^{\frac{2d}{d-1}} \,\mathrm{d}y \Big)^{\frac{d-1}{2d}} \\ \leq C \bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,2r}} |\nabla u|^2 + r^{-2}\Big(1 + |r^2\lambda|\Big)|u|^2 \,\mathrm{d}y \bigg)^{\frac{1}{2}} \\ \leq C \bigg(\frac{1}{r^d} \int_{\Omega \cap B(x_0,\alpha_1 r)} |\nabla u|^2 + r^{-2}\Big(1 + |r^2\lambda|\Big)|u|^2 \,\mathrm{d}y \bigg)^{\frac{1}{2}},\end{aligned}$$

with $\alpha_1 := [4 + (20d(M+1)^2)]^{1/2}$. Next, use Caccioppoli's inequality, see [88, Lem. 2.1], to deduce

$$|\lambda| \int_{\Omega \cap B(x_0,\alpha_1 r)} |u|^2 \,\mathrm{d}x \leq \frac{C}{r^2} \int_{\Omega \cap B(x_0,2\alpha_1 r)} |u|^2 \,\mathrm{d}x$$

with a constant $C > 0$ depending only on d, θ, and the ellipticity constant. Moreover, Poincaré's inequality yields a constant $C > 0$ depending only on d and M such that

$$\int_{\Omega \cap B(x_0,2\alpha_1 r)} |u|^2 \,\mathrm{d}x \leq C r^2 \int_{\Omega \cap B(x_0,2\alpha_1 r)} |\nabla u|^2 \,\mathrm{d}x.$$

This gives the desired weak reverse Hölder estimate.

(5) Finally, we conclude the gradient estimate of the resolvent in L^p by means of the self-improving property of weak reverse Hölder estimates, Proposition 3.1.4, and the L^p-extrapolation theorem, Theorem 3.1.2.

Let us see, whether we can perform the same steps for the Stokes resolvent problem. Concerning (1), note that the Stokes resolvent problem has the following scaling behavior

$$\begin{aligned} & \lambda u - \Delta u + \nabla \phi = 0 && \text{in } \Omega \cap U_{x_0,3r} \\ \Leftrightarrow \qquad & r^2 \lambda u_r - \Delta u_r + \nabla \phi_r = 0 && \text{in } U_{x_0,3}^{\mathrm{resc},+}, \end{aligned}$$

with $u_r(y) := u(ry)$ and $\phi_r(y) := r\phi(ry)$.

The second point, (2), is also no problem as the proof for the Stokes resolvent problem served as the reference for the proof in the elliptic situation.

In (3), we can appeal to the regularity theory of the Stokes resolvent problem, Theorem 4.1.9, which was resolved in Section 4.3. The estimate of the non-tangential maximal function in L^2 by the boundary data differs from the one used in (3) and reads in our situation

$$\begin{aligned} \|(\nabla u_r)^*\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+})} \leq C \Big\{ & \|\nabla_{\tan} u_r\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega; \mathbb{C}^{d^2})} \\ & + |r^2 \lambda|^{\frac{1}{2}} \|u_r\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega; \mathbb{C}^d)} \\ & + |r^2\lambda| \|\langle \nu, u_r \rangle\|_{\mathrm{W}^{1,2}(\partial U_{x_0,s}^{\mathrm{resc},+})^*} \Big\}. \end{aligned}$$

To handle the additional term, use the trivial estimate by controlling u_r in the $(\mathrm{W}^{1,2}(\partial U_{x_0,s}^{\mathrm{resc},+}))^*$-norm by its $\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+})$-norm. Note that there is no contribution to the integral on the portion of $\partial U_{x_0,s}^{\mathrm{resc},+}$, which comes from $r^{-1}\partial\Omega$, since u_r vanishes on that part. It follows that

$$\|\langle \nu, u_r \rangle\|_{\mathrm{W}^{1,2}(\partial U_{x_0,s}^{\mathrm{resc},+})^*} \leq \|u_r\|_{\mathrm{L}^2(\partial U_{x_0,s}^{\mathrm{resc},+} \setminus r^{-1}\partial\Omega; \mathbb{C}^d)}. \tag{5.16}$$

Next, perform the integration argument in (4) with this modified estimate for $(\nabla u_r)^*$ to deduce that

$$\begin{aligned} & \bigg(\int_{U_{x_0,1}^{\mathrm{resc},+}} |\nabla u_r|^{\frac{2d}{d-1}} \, \mathrm{d}y \bigg)^{\frac{d-1}{2d}} \\ & \qquad \leq C \bigg(\int_{U_{x_0,2}^{\mathrm{resc},+}} |\nabla u_r|^2 + |r^2\lambda| \Big(1 + |r^2\lambda|^{\frac{1}{2}}\Big)^2 |u_r|^2 \, \mathrm{d}y \bigg)^{\frac{1}{2}}. \end{aligned}$$

As in the elliptic situation use $(1 + |r^2\lambda|^{\frac{1}{2}})^2 \leq 2 + 2|r^2\lambda|$, the linear transformation $x = ry$, and $U_{x_0,2r} \subset B(x_0, \alpha_1 r)$ to derive

$$
\begin{aligned}
&\Big(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}} |\nabla u|^{\frac{2d}{d-1}}\,\mathrm{d}x\Big)^{\frac{d-1}{2d}} \\
&\quad\leq C\Big(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha_1 r)} |\nabla u|^2 + r^{-2}\Big(|r^2\lambda| + |r^2\lambda|^2\Big)|u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}
\end{aligned}
\tag{5.17}
$$

In the elliptic situation, we would now appeal to Caccioppoli's inequality. For the Stokes resolvent problem, Caccioppoli's inequality has the following form.

Proposition 5.3.2 (Caccioppoli). *Let $\theta \in [0, \pi)$, $\lambda \in \mathrm{S}_\theta$, $x_0 \in \overline{\Omega}$, and $r > 0$. Let further $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$ with $f = 0$ on $\Omega \cap B(x_0, 2r)$, $u := (\lambda + A_2)^{-1}\mathbb{P}_2 f$, and $\pi \in \mathrm{L}^2(\Omega)$ be the pressure associated to u. Then there exists a constant $C > 0$ depending only on d and θ such that*

$$
\begin{aligned}
&|\lambda|\int_{\Omega\cap B(x_0,r)} |u|^2\,\mathrm{d}x + \int_{\Omega\cap B(x_0,r)} |\nabla u|^2\,\mathrm{d}x \\
&\quad\leq \frac{C}{r^2}\Big\{\frac{1}{|\lambda|}\int_{\Omega\cap B(x_0,2r)} |\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2\,\mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |u|^2\,\mathrm{d}x\Big\}.
\end{aligned}
$$

Proof. We imitate the standard proof of Caccioppoli's inequality for the Laplacian, see, e.g., Giaquinta and Martinazzi [37, Thm. 4.1].

Let $\varphi \in \mathrm{C}_c^\infty(B(x_0, 2r))$ be a smooth cut-off function with $\varphi = 1$ on $B(x_0, r)$, $0 \leq \varphi \leq 1$, and $\|\nabla\varphi\|_{\mathrm{L}^\infty(B(x_0,2r);\mathbb{R}^d)} \leq C_d/r$, where $C_d > 0$ is a constant depending solely on d. By smoothness of φ, it follows that $\psi := \varphi^2 u \in \mathrm{W}_0^{1,2}(\Omega; \mathbb{C}^d)$. Appealing to Theorem 5.2.3 it follows by testing with ψ

$$
\begin{aligned}
&\lambda\int_{\Omega\cap B(x_0,2r)} |\varphi u|^2\,\mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} \langle\nabla u, \nabla[\varphi^2 u]\rangle\,\mathrm{d}x \\
&\qquad - \int_{\Omega\cap B(x_0,2r)} \pi\operatorname{div}(\varphi^2\overline{u})\,\mathrm{d}x = \int_{\Omega\cap B(x_0,2r)} \langle\mathbb{P}_2 f, \varphi^2 u\rangle\,\mathrm{d}x.
\end{aligned}
$$

First, by means of the product rule

$$
\begin{aligned}
&\int_{\Omega\cap B(x_0,2r)} \langle\nabla u, \nabla[\varphi^2 u]\rangle\,\mathrm{d}x \\
&\quad = \int_{\Omega\cap B(x_0,2r)} \langle\varphi\nabla u, \nabla[\varphi u]\rangle\,\mathrm{d}x + \sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} \langle\varphi\nabla u_j, u_j\nabla\varphi\rangle\,\mathrm{d}x.
\end{aligned}
$$

Next, rewrite $\varphi\nabla u$ and $\varphi\nabla u_j$ using the product rule

$$
\begin{aligned}
&= \int_{\Omega\cap B(x_0,2r)} |\nabla\varphi u|^2 \, \mathrm{d}x - \sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} \langle u_j \nabla\varphi, \nabla[\varphi u_j]\rangle \, \mathrm{d}x \\
&\quad + \sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} \langle \nabla[\varphi u_j], u_j\nabla\varphi\rangle \, \mathrm{d}x - \int_{\Omega\cap B(x_0,2r)} |\nabla\varphi|^2 |u|^2 \, \mathrm{d}x.
\end{aligned}
$$

Second, rewrite by means of Lemma 5.1.3 and $f = 0$ on $\Omega\cap B(x_0, 2r)$

$$
\int_{\Omega\cap B(x_0,2r)} \langle \mathbb{P}_2 f, \varphi^2 u\rangle \, \mathrm{d}x = \int_{\Omega\cap B(x_0,2r)} \langle \nabla(-\Delta_N)\operatorname{div}(f), \varphi^2 u\rangle \, \mathrm{d}x.
$$

Next, use integration by parts and $\operatorname{div}(\varphi^2\overline{u}) = 2\varphi\langle\nabla\varphi, u\rangle$ to derive

$$
= -2\int_{\Omega\cap B(x_0,2r)} (-\Delta_N)\operatorname{div}(f)\langle\nabla\varphi, \varphi u\rangle \, \mathrm{d}x.
$$

Third, use the same identity for $\operatorname{div}(\varphi^2\overline{u})$ to get

$$
\int_{\Omega\cap B(x_0,2r)} \pi \operatorname{div}(\varphi^2\overline{u}) \, \mathrm{d}x = 2\int_{\Omega\cap B(x_0,2r)} \pi\langle\nabla\varphi, \varphi u\rangle \, \mathrm{d}x.
$$

Finally, we plug all identities into the very first equation, to deduce that

$$
\begin{aligned}
&\lambda \int_{\Omega\cap B(x_0,2r)} |\varphi u|^2 \, \mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x \\
&\quad = 2\sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} \operatorname{Re}(\langle u_j\nabla\varphi, \nabla[\varphi u_j]\rangle) \, \mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |\nabla\varphi|^2|u|^2 \, \mathrm{d}x \\
&\qquad + 2\int_{\Omega\cap B(x_0,2r)} [\pi - (-\Delta_N)\operatorname{div}(f)]\langle\nabla\varphi, \varphi u\rangle \, \mathrm{d}x.
\end{aligned}
$$

Take the absolute value of this identity and note that the left-hand side is estimated from below by means of Lemma 5.2.4

$$
\begin{aligned}
&|\lambda| \int_{\Omega\cap B(x_0,2r)} |\varphi u|^2 \, \mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x \\
&\qquad \leq C_\theta \Big| \lambda \int_{\Omega\cap B(x_0,2r)} |\varphi u|^2 \, \mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x \Big|,
\end{aligned}
$$

where $C_\theta > 0$ depends only on θ. Estimate each term on the right-hand side separately by using first $0 \leq \varphi \leq 1$ and $\|\nabla\varphi\|_{\mathrm{L}^\infty(B(x_0,2r);\mathbb{R}^d)} \leq C_d/r$, and then Young's inequality

$$\begin{aligned}
&\Big| 2 \sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} \operatorname{Re}(\langle u_j \nabla\varphi, \nabla[\varphi u_j]\rangle)\, \mathrm{d}x \Big| \\
&\qquad \leq \frac{2C_d}{r} \sum_{j=1}^d \int_{\Omega\cap B(x_0,2r)} |u_j|\, |\nabla[\varphi u_j]|\, \mathrm{d}x \\
&\qquad \leq \frac{2C_\theta C_d^2}{r^2} \int_{\Omega\cap B(x_0,2r)} |u|^2 \, \mathrm{d}x + \frac{1}{2C_\theta} \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Big| 2 \int_{\Omega\cap B(x_0,2r)} [\pi - (-\Delta_N)\operatorname{div}(f)] \langle \nabla\varphi, \varphi u\rangle \, \mathrm{d}x \Big| \\
&\leq \frac{2C_\theta C_d^2}{|\lambda|\, r^2} \int_{\Omega\cap B(x_0,2r)} |\pi - (-\Delta_N)\operatorname{div}(f)|^2 \, \mathrm{d}x + \frac{|\lambda|}{2C_\theta} \int_{\Omega\cap B(x_0,2r)} |\varphi u|^2 \, \mathrm{d}x.
\end{aligned}$$

Finally,

$$\int_{\Omega\cap B(x_0,2r)} |\nabla\varphi|^2 |u|^2 \, \mathrm{d}x \leq \frac{C_d^2}{r^2} \int_{\Omega\cap B(x_0,2r)} |u|^2 \, \mathrm{d}x.$$

Now, absorb the terms

$$\frac{1}{2C_\theta} \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x \quad \text{and} \quad \frac{1}{2C_\theta} \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x$$

to the left-hand side, to deduce

$$\begin{aligned}
&\frac{|\lambda|}{2} \int_{\Omega\cap B(x_0,2r)} |\varphi u|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\Omega\cap B(x_0,2r)} |\nabla[\varphi u]|^2 \, \mathrm{d}x \\
&\leq \frac{2C_\theta^2 C_d^2}{r^2} \Big\{ \frac{1}{|\lambda|} \int_{\Omega\cap B(x_0,2r)} |\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2 \, \mathrm{d}x + \int_{\Omega\cap B(x_0,2r)} |u|^2 \, \mathrm{d}x \Big\}.
\end{aligned}$$

We conclude the proof by means of $\varphi = 1$ on $B(x_0, r)$. $\square$

Recall our intermediate result for the weak reverse Hölder estimates in (5.17). Take the term with the highest exponent in λ and apply Caccioppoli's inequality to obtain

$$|r^2\lambda|^2 \int_{\Omega\cap B(x_0,\alpha_1 r)} |u|^2 \,\mathrm{d}x \leq C\Big\{ r^2 \int_{\Omega\cap B(x_0,2\alpha_1 r)} |\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2 \,\mathrm{d}x + |r^2\lambda| \int_{\Omega\cap B(x_0,2\alpha_1 r)} |u|^2 \,\mathrm{d}x \Big\}.$$

Plugging this inside (5.17), we arrive at

$$\Big(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}} |\nabla u|^{\frac{2d}{d-1}} \,\mathrm{d}x\Big)^{\frac{d-1}{2d}} \leq C\Big(\frac{1}{r^d}\int_{\Omega\cap B(x_0,2\alpha_1 r)} |\nabla u|^2 + ||\lambda|^{\frac{1}{2}} u|^2 + |\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{5.18}$$

Now, we are at the point, where we might have to ask ourselves, whether this is the best right-hand side, we can get. In the elliptic situation, we treated the term $||\lambda|^{\frac{1}{2}} u|^2$ with Caccioppoli's inequality as well. But what would be the effect of that? This would result in two more terms in the integral, namely

$$r^{-2}\,|\lambda|^{-1}\,|\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2 \quad \text{and} \quad r^{-2}|u|^2.$$

While the second term is not a problem at all, since this term can easily be transformed into a $|\nabla u|^2$ by means of Poincaré's inequality, the first term results in a second term that involves the pressure with a different prefactor. In the end of this discussion, we will see that the factor $|\lambda|^{-1}$ is very desirable. However, factor r^{-2} does not fit into the weak reverse Hölder estimates framework, as they dictate the exact exponent of the radius r. Moreover, as the term $|\pi - (\Delta_N)^{-1}\operatorname{div}(f)|^2$ appears in the integral anyhow, we conclude that the right-hand side above is the best we can achieve.

The thought is now, that the right-hand side of (5.18) may be the right-hand side of a weak reverse Hölder estimate of the function

$$x \mapsto \Big[|\nabla u(x)|^2 + |\pi(x) - [(\Delta_N)^{-1}\operatorname{div}(f)](x)|^2 + ||\lambda|^{\frac{1}{2}} u(x)|^2\Big]^{\frac{1}{2}}.$$

We will prove in the following, that this is indeed the case. For this purpose, we abbreviate the term that involves the pressure by

$$\phi := \pi - (\Delta_N)^{-1} \operatorname{div}(f). \tag{5.19}$$

To proceed, note that by [89, Lem. 6.1], the function u satisfies the weak reverse Hölder estimate

$$\Big(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}} \big||\lambda|^{\frac{1}{2}}\, u\big|^{\frac{2d}{d-1}}\,\mathrm{d}x\Big)^{\frac{d-1}{2d}} \leq C\Big(\frac{1}{r^d}\int_{\Omega\cap B(x_0,2\alpha_1 r)} \big||\lambda|^{\frac{1}{2}}\, u\big|^2\,\mathrm{d}x\Big)^{\frac{1}{2}},$$

which trivially results in the estimate

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}} \big||\lambda|^{\frac{1}{2}}\, u\big|^{\frac{2d}{d-1}}\,\mathrm{d}x\Big)^{\frac{d-1}{2d}}& \\ \leq C\Big(\frac{1}{r^d}\int_{\Omega\cap B(x_0,2\alpha_1 r)} &|\nabla u|^2 + \big||\lambda|^{\frac{1}{2}} u\big|^2 + |\phi|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.20}$$

It remains to estimate ϕ. This calculation will be a little bit lenghty, but in essence, we trace the constants of the well-known pressure estimate in order to reveal the inherent dependence of r. This results in the fact, that $|\phi|$ is controlled by $||\lambda|^{\frac{1}{2}} u|$ and $|\nabla u|$.

Note that due to Lemma 5.1.3 and Theorem 5.2.3, we find that for all $v \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^d)$

$$\lambda\int_\Omega \langle u, v\rangle\,\mathrm{d}x + \int_\Omega \langle\nabla u, \nabla v\rangle\,\mathrm{d}x - \int_\Omega \phi\overline{\operatorname{div}(v)}\,\mathrm{d}x = \int_\Omega \langle f, v\rangle\,\mathrm{d}x.$$

To start, recall that $\Omega \cap U_{x_0,r}$ is the rotation and a translation of the Lipschitz cylinder

$$D_{\eta_{x_0}}(r) = \{(x', x_d) : |x'| < r, \eta_{x_0}(x') < x_d < 10d(M+1)r\}.$$

Here η_{x_0} is a Lipschitz continuous function with $\eta_{x_0}(0) = 0$ and Lipschitz constant bounded by M. We need in the following, that this set is star-shaped with respect to a ball, whose radius is comparable to r. Let $x \in D_{\eta_{x_0}}(r)$ with $x_d > 2Mr$ and $y' \in \mathbb{R}^{d-1}$ with $|y'| < r$. Note that the Lipschitz continuity together with $\eta_{x_0}(0) = 0$ implies that $|\eta_{x_0}(y')| < Mr$

for every $|y'| < r$. We claim that the line connecting x and $(y', \eta_{x_0}(y'))$ does not contain another point of the graph of η_{x_0}. Indeed, assume that for some $s \in (0,1)$, we have

$$\begin{aligned} & (z', \eta_{x_0}(z')) = sx + (1-s)(y', \eta_{x_0}(y')) \\ \Leftrightarrow \quad & (z', \eta_{x_0}(z')) - (y', \eta_{x_0}(y')) = s[x - (y', \eta_{x_0}(y')] \end{aligned}$$

for some $|z'| < r$. This implies that

$$\frac{|\eta_{x_0}(z') - \eta_{x_0}(y')|}{|z'-y'|} = \frac{|x_d - \eta_{x_0}(y')|}{|x'-y'|}.$$

By Lipschitz continuity, we derive for the left-hand side

$$\frac{|\eta_{x_0}(z') - \eta_{x_0}(y')|}{|z'-y'|} \leq M$$

and since $x_d > 2Mr$, we find together with $|\eta_{x_0}(y')| < Mr$ for the right-hand side

$$\frac{|x_d - \eta_{x_0}(y')|}{|x'-y'|} > \frac{2Mr - Mr}{|x'-y'|} \geq M.$$

This is a contradiction. It follows that $D_{\eta_{x_0}}(r)$ is star-shaped with respect to every point x that satisfies $2Mr < x_d < 10d(M+1)r$. In particular, it is star-shaped with respect to a ball of radius r. Because $U_{x_0,r}$ is a rotation and a translation of $D_{\eta_{x_0}}(r)$, the same is valid for $\Omega \cap U_{x_0,r}$.

Next, define $p := 2d/(d-1)$ and let $\mathcal{B} : \mathrm{L}^{p'}_0(\Omega \cap U_{x_0,r}) \to \mathrm{W}^{1,p'}_0(\Omega \cap U_{x_0,r})$ be Bogovskiĭ's operator, i.e., it satisfies $\operatorname{div}(\mathcal{B}f) = f$ and

$$\|\nabla \mathcal{B} f\|_{\mathrm{L}^{p'}(\Omega \cap U_{x_0,r})} \leq C \|f\|_{\mathrm{L}^{p'}(\Omega \cap U_{x_0,r})} \qquad (f \in \mathrm{L}^{p'}(\Omega \cap U_{x_0,r})).$$

Such an operator is constructed in Galdi [34, Lem. 3.1] and it was proven that in our particular situation, C depends only on d. Let $\phi_{\Omega \cap U_{x_0,r}}$ denote the mean value of ϕ on $\Omega \cap U_{x_0,r}$. Then, calculate by duality

$$
\begin{aligned}
\bigg(\int_{\Omega \cap U_{x_0,r}} & |\phi - \phi_{\Omega \cap U_{x_0,r}}|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} \\
&= \sup_{\substack{g \in \mathrm{L}^{p'}_0 (\Omega \cap U_{x_0,r}) \\ \|g\|_{\mathrm{L}^{p'}} \leq 1}} \bigg| \int_{\Omega \cap U_{x_0,r}} [\phi - \phi_{\Omega \cap U_{x_0,r}}] \overline{g} \, \mathrm{d}x \bigg| \\
&= \sup_{\substack{g \in \mathrm{L}^{p'}_0 (\Omega \cap U_{x_0,r}) \\ \|g\|_{\mathrm{L}^{p'}} \leq 1}} \bigg| \int_{\Omega \cap U_{x_0,r}} [\phi - \phi_{\Omega \cap U_{x_0,r}}] \overline{\operatorname{div}(\mathcal{B} g)} \, \mathrm{d}x \bigg| \\
&\leq \sup_{\substack{g \in \mathrm{L}^{p'}_0 (\Omega \cap U_{x_0,r}) \\ \|g\|_{\mathrm{L}^{p'}} \leq 1}} \bigg| \lambda \int_{\Omega \cap U_{x_0,r}} \langle u, \mathcal{B} g \rangle \, \mathrm{d}x \bigg| \\
&\qquad + \sup_{\substack{g \in \mathrm{L}^{p'}_0 (\Omega \cap U_{x_0,r}) \\ \|g\|_{\mathrm{L}^{p'}} \leq 1}} \bigg| \int_{\Omega \cap U_{x_0,r}} \langle \nabla u, \nabla \mathcal{B} g \rangle \, \mathrm{d}x \bigg|.
\end{aligned}
$$

By Hölder's inequality, followed by Poincaré's inequality and the estimate on $\nabla \mathcal{B}$, we derive

$$
\leq C \bigg\{ r \, |\lambda|^{\frac{1}{2}} \bigg(\int_{\Omega \cap U_{x_0,r}} ||\lambda|^{\frac{1}{2}} u|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} + \bigg(\int_{\Omega \cap U_{x_0,r}} |\nabla u|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} \bigg\}.
$$

Altogether, we obtain

$$
\begin{aligned}
\bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} & |\phi|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} \\
&\leq \bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\phi - \phi_{\Omega \cap U_{x_0,r}}|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} + \frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\phi| \, \mathrm{d}x \\
&\leq C \bigg\{ r \, |\lambda|^{\frac{1}{2}} \bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} ||\lambda|^{\frac{1}{2}} u|^p \bigg)^{\frac{1}{p}} + \bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\nabla u|^p \bigg)^{\frac{1}{p}} \\
&\qquad + \frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\phi| \, \mathrm{d}x \bigg\}.
\end{aligned}
$$

Note that by means of Hölder's inequality,

$$
\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\phi| \, \mathrm{d}x \leq C \bigg(\frac{1}{r^d} \int_{\Omega \cap U_{x_0,r}} |\phi|^2 \, \mathrm{d}x \bigg)^{\frac{1}{2}},
$$

where C depends only on d and M. Moreover, estimate by the known weak reverse Hölder estimates on u

$$r\,|\lambda|^{\frac{1}{2}}\left(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}}||\lambda|^{\frac{1}{2}}\,u|^p\right)^{\frac{1}{p}}\leq Cr\,|\lambda|^{\frac{1}{2}}\left(\frac{1}{r^d}\int_{\Omega\cap B(x_0,\alpha_1 r)}||\lambda|^{\frac{1}{2}}\,u|^2\right)^{\frac{1}{2}}.$$

Finally, appeal to Caccioppoli's inequality, to conclude

$$\leq C\biggl(\frac{1}{r^d}\int_{\Omega\cap B(x_0,2\alpha_1 r)}|\phi|^2+||\lambda|^{\frac{1}{2}}u|^2\,\mathrm{d}x\biggr)^{\frac{1}{2}}.$$

Summarizing this estimate for the term involving the pressure together with (5.18), (5.19), and (5.20), we have established the weak reverse Hölder estimate

$$\begin{aligned}&\biggl(\frac{1}{r^d}\int_{\Omega\cap U_{x_0,r}}\Bigl[|\nabla u|^2+||\lambda|^{\frac{1}{2}}u|^2+|\pi-(\Delta_N)^{-1}\operatorname{div}(f)|^2\Bigr]^{\frac{d}{d-1}}\,\mathrm{d}x\biggr)^{\frac{d-1}{2d}}\\&\qquad\leq C\biggl(\frac{1}{r^d}\int_{\Omega\cap B(x_0,2\alpha_1 r)}|\nabla u|^2+||\lambda|^{\frac{1}{2}}u|^2+|\pi-(\Delta_N)^{-1}\operatorname{div}(f)|^2\,\mathrm{d}x\biggr)^{\frac{1}{2}}.\end{aligned}$$

Due to the following proposition, the interior weak reverse Hölder estimates hold as well.

Proposition 5.3.3. *Let $x_0\in\Omega$ and $r>0$ such that $B(x_0,3r)\subset\Omega$. Let $\lambda\in\mathrm{S}_\theta$, $f\in\mathrm{L}^2(\Omega;\mathbb{C}^d)$ with $f=0$ on $B(x_0,3r)$, $u:=(\lambda+A_2)^{-1}\mathbb{P}_2 f$ and π be the pressure associated to u via Theorem 5.2.3. Then there exists a constant $C>0$ depending only on d and θ such that*

$$\biggl(\frac{1}{r^d}\int_{B(x_0,r)}|\nabla u|^{\frac{2d}{d-1}}\,\mathrm{d}x\biggr)^{\frac{d-1}{2d}}\leq C\biggl(\frac{1}{r^d}\int_{B(x_0,2r)}|\nabla u|^2\,\mathrm{d}x\biggr)^{\frac{1}{2}}$$

and

$$\begin{aligned}&\biggl(\frac{1}{r^d}\int_{B(x_0,r)}|\pi-(-\Delta_N)^{-1}\operatorname{div}(f)|^{\frac{2d}{d-1}}\,\mathrm{d}x\biggr)^{\frac{d-1}{2d}}\\&\qquad\leq C\biggl(\frac{1}{r^d}\int_{B(x_0,2r)}|\pi-(-\Delta_N)^{-1}\operatorname{div}(f)|^2\,\mathrm{d}x\biggr)^{\frac{1}{2}}.\end{aligned}$$

Proof. Note that by virtue of Theorem 5.2.3, Lemma 5.1.3, and $f = 0$ on $B(x_0, 2r)$, u and $\pi - (-\Delta_N)^{-1} \operatorname{div}(f)$ solve for all $v \in \mathrm{C}_c^\infty(B(x_0, 3r))$

$$\lambda \int_{B(x_0,3r)} \langle u, v\rangle \, \mathrm{d}x + \int_{B(x_0,3r)} \langle \nabla u, \nabla v\rangle \, \mathrm{d}x \\ - \int_{B(x_0,3r)} [\pi - (-\Delta_N)^{-1} \operatorname{div}(f)] \overline{\operatorname{div}(v)} \, \mathrm{d}x = 0.$$

By interior regularity, see Galdi [34, Thm. IV.4.2], u and $\phi := \pi - (-\Delta_N)^{-1} \operatorname{div}(f)$ are smooth solutions of

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = 0 & \text{in } B(x_0, 3r) \\ \operatorname{div}(u) = 0 & \text{in } B(x_0, 3r). \end{cases}$$

Taking the divergence of the first equation shows that ϕ is harmonic. Now, the weak reverse Hölder estimate for ϕ is a consequence of de Giorgi's theorem, see, e.g., Giaquinta and Martinazzi [37, Thm. 8.13].

The weak reverse Hölder estimate for ∇u can be derived from the weak reverse Hölder estimates for u. Indeed, by Schwarz' theorem, also $v := \partial_i u$ and $\psi := \partial_i \phi$ solve the homogeneous Stokes resolvent problem in $B(x_0, 2r)$. Thus, appealing to Shen [89, Lem. 6.2], we find

$$\begin{aligned} \Big(\frac{1}{r^d} \int_{B(x_0,r)} |\partial_i u|^{p_\varepsilon} \, \mathrm{d}x \Big)^{\frac{1}{p_\varepsilon}} &\leq C \Big(\frac{1}{r^d} \int_{B(x_0,2r)} |\partial_i u|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\ &\leq C \Big(\frac{1}{r^d} \int_{B(x_0,2r)} |\nabla u|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}. \end{aligned}$$

Doing this for every $1 \leq i \leq d$ concludes the proof. □

A consequence of the preceding discussion is that we have found a linear operator, which satisfies weak reverse Hölder estimates in the interior and on the boundary. Moreover, all implicit constants are uniform in $\lambda \in \mathrm{S}_\theta$. This operator is defined as

$$T_\lambda : \mathrm{L}^2(\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\Omega; \mathbb{C}^{d^2+d+1}), \quad f \mapsto \begin{pmatrix} |\lambda|^{\frac{1}{2}} \nabla (\lambda + A_2)^{-1} \mathbb{P}_2 f \\ |\lambda| \, (\lambda + A_2)^{-1} \mathbb{P}_2 f \\ |\lambda|^{\frac{1}{2}} \, [\pi - (-\Delta_N)^{-1} \operatorname{div}(f)] \end{pmatrix}.$$

Note that we multiplied everything with $|\lambda|^{\frac{1}{2}}$, which does no harm to the uniformity of the constants of the weak reverse Hölder estimates in λ. Furthermore, note that the correspondence between $(\lambda + A_2)^{-1}\mathbb{P}_2 f$ and π from Theorem 5.2.3 is linear, so that T_λ is indeed linear. The boundedness of this family in $\mathcal{L}(\mathrm{L}^p(\Omega; \mathbb{C}^d), \mathrm{L}^p(\Omega; \mathbb{C}^{d^2+d+1}))$ would imply the boundedness of the family $\{|\lambda|^{\frac{1}{2}} \nabla(\lambda + A_p)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ since

$$\|T_\lambda f\|_{\mathrm{L}^p(\Omega; \mathbb{C}^{d^2+d+1})} \geq |\lambda|^{\frac{1}{2}} \|\nabla(\lambda + A_2)^{-1}\mathbb{P}_2 f\|_{\mathrm{L}^p(\Omega; \mathbb{C}^{d^2})}.$$

To conclude the boundedness of $\{T_\lambda\}_{\lambda \in \mathrm{S}_\theta}$ in $\mathcal{L}(\mathrm{L}^p(\Omega; \mathbb{C}^d), \mathrm{L}^p(\Omega; \mathbb{C}^{d^2+d+1}))$, we only need its boundedness in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^d), \mathrm{L}^2(\Omega; \mathbb{C}^{d^2+d+1}))$, because then, we could directly appeal to Theorem 3.1.2. As the boundedness of $|\lambda|^{\frac{1}{2}} \nabla(\lambda + A_2)^{-1}\mathbb{P}_2$ and $|\lambda| (\lambda + A_2)^{-1}\mathbb{P}_2$ is proven in Proposition 5.2.5, we only need the boundedness of the family that involves the pressure. Unfortunately, the following lemma is valid.

Lemma 5.3.4. *Assume that there exists a constant $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ and all $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$*

$$|\lambda|^{\frac{1}{2}} \|\pi - (-\Delta_N)^{-1} \operatorname{div}(f)\|_{\mathrm{L}^2(\Omega)} \leq C\|f\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)}.$$

Then, the spaces $\mathrm{L}^2_\sigma(\Omega)$ and $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ coincide.

Proof. Let $T : \mathrm{L}^2(\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\Omega)$ be the operator that maps f onto $\pi - (-\Delta_N)^{-1} \operatorname{div}(f)$ and fix $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$.

Since π is the pressure associated to $u := (\lambda + A_2)^{-1}\mathbb{P}_2 f$ and since $\mathbb{P}_2$ is a projection, we see that π is also the pressure associated to $(\lambda + A_2)^{-1}\mathbb{P}_2\mathbb{P}_2 f$. Moreover, because $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$, we see that

$$T\mathbb{P}_2 f = \pi.$$

By assumption

$$\|T\mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega)} \leq C\, |\lambda|^{-\frac{1}{2}} \|\mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)},$$

This, together with the assumption yields

$$\begin{aligned}\|(-\Delta_N)^{-1} \operatorname{div}(f)\|_{\mathrm{L}^2(\Omega)} &\leq \|(-\Delta_N)^{-1} \operatorname{div}(f) - \pi\|_{\mathrm{L}^2(\Omega)} + \|\pi\|_{\mathrm{L}^2(\Omega)} \\ &\leq C\, |\lambda|^{-\frac{1}{2}} \Big\{\|f\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)} + \|\mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)}\Big\}.\end{aligned}$$

Letting $|\lambda| \to \infty$, we conclude that $(-\Delta_N)^{-1} \operatorname{div}(f) = 0$. An application of Lemma 5.1.3 shows that $\mathrm{L}^2_\sigma(\Omega)$ and $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ coincide. $\square$

As the spaces $\mathrm{L}^2_\sigma(\Omega)$ and $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ do certainly not coincide, the family $\{T_\lambda\}_{\lambda \in \mathrm{S}_\theta}$ cannot be bounded in $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbb{C}^d), \mathrm{L}^2(\Omega; \mathbb{C}^{d^2+d+1}))$. Note that this non-uniform boundedness arises from the large exponent of $|\lambda|$ in front of the pressure term. If this exponent would be zero, i.e., if there would be no $|\lambda|$ in front of the pressure term at all, there would be no problem, since one can easily prove that

$$f \mapsto \pi - (-\Delta_N)^{-1} \operatorname{div}(f)$$

gives rise to a uniformly bounded family on L^2. Finally, note that this large exponent originated in (5.16). If this estimate could be improved, there would be a chance to establish the gradient estimates of the Stokes resolvent family, and hence, of the Stokes semigroup.

CHAPTER 6

Existence results for the Navier-Stokes equations

In this chapter, we will present two approaches to the existence of solutions of the Navier-Stokes equations in three dimensional bounded Lipschitz domains Ω

$$\text{(NSE)} \qquad \begin{cases} \partial_t u - \Delta u + \langle u, \nabla \rangle u + \nabla \pi = f & \text{in } \Omega, t > 0 \\ \operatorname{div}(u) = 0 & \text{in } \Omega, t > 0 \\ u(0) = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, t > 0. \end{cases}$$

Here, the vector field $u : [0, \infty) \times \Omega \to \mathbb{C}^3$ describes the velocity field of an incompressible fluid, $\pi : [0, \infty) \times \Omega \to \mathbb{C}$ its pressure, and $f : [0, \infty) \times \Omega \to \mathbb{C}^3$ an external force. The third equation gives the initial configuration a, i.e., the velocity field at time $t = 0$ in Ω.

The first approach that is presented exploits the maximal L^q-regularity of the Stokes operator A_p on $\mathrm{L}^p_\sigma(\Omega)$ as well as embeddings of $\mathcal{D}(A_p)$ into Bessel potential spaces in order to perform a fixed point argument. This argument was used several times in the literature in order to deduce the existence of strong solutions of the Navier-Stokes equations, see, e.g., Solonnikov [91] and Saal [84].

The second approach is based upon an iteration scheme, which is used by Giga [41] to derive the existence of local mild solutions in the critical space

$\mathrm{BC}([0,T_0);\mathrm{L}^3_\sigma(\Omega))$ for any initial conditions of arbitrary size and of global mild solutions in $\mathrm{BC}([0,\infty);\mathrm{L}^3_\sigma(\Omega))$ for initial conditions with sufficiently small $\mathrm{L}^3_\sigma(\Omega)$-norm. Here, BC stands for bounded and continuous. On the whole space this is a classical result due to KATO, see [59], who uses the same iteration scheme. However, KATO the convergence of the iteration scheme differently than GIGA, and needs gradient estimates for the Stokes-semigroup on $\mathrm{L}^3_\sigma(\Omega)$ and additional L^p-L^q-estimates of the semigroup. We have seen in Section 5.3, that the gradient estimates are not available at the present time. But still, some modifications in the proof of GIGA show that the weaker estimates proven in Theorem 5.2.22 suffice in order to establish the existence result above. A comparison of GIGA'S and KATO'S proofs reveals that GIGA exploits the divergence-form structure in which the nonlinearity of the Navier-Stokes equations can be rewritten. This structural information is neglected by KATO and seems to be the reason for the need of stronger estimates compared to the proof of GIGA.

6.1 The main results of this chapter

We start with a notion of solutions that arises by employing the approach via maximal L^q-regularity.

Definition 6.1.1. Let $1 < q < \infty$ and let $2 \leq p < 3 + \varepsilon$, with $\varepsilon > 0$ from Convention 5.2.25. Given $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))$ and $a \in \mathrm{L}^p_\sigma(\Omega)$, the functions (u,π) are called *weak (p,q)-solutions to* (NSE) if

$$u \in \mathrm{W}^{1,q}(0,\infty;\mathrm{L}^p_\sigma(\Omega)) \cap \mathrm{L}^q(0,\infty;\mathcal{D}(A_p)), \quad \pi(t) \in \mathrm{L}^p(\Omega) \quad (\text{a.e. } t > 0),$$

if for all $\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^3)$ and almost every $t > 0$

$$\int_\Omega \langle \partial_t u(t) + \langle u(t), \nabla\rangle u(t), \varphi\rangle + \langle \nabla u(t), \nabla\varphi\rangle - \pi(t)\,\mathrm{div}(\varphi)\,\mathrm{d}x = \int_\Omega \langle f(t),\varphi\rangle\,\mathrm{d}x,$$

and if the initial condition a is realized in the sense that

$$\|u(t) - a\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \to 0 \quad \text{as} \quad t \searrow 0.$$

Remark 6.1.2. (1) Note that the discussion in Subsection 1.2.2 and especially Proposition 1.2.7 imply that $a \in (\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q,q}$ if (u, π) are weak (p, q)-solutions corresponding to an initial velocity field a.

(2) We must check that the distributional formulation even makes sense, i.e., that $\langle u(t), \nabla \rangle u(t)$ is in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ for almost every $t > 0$. For this, note that $u(t) \in \mathcal{D}(A_p)$ for almost every $t > 0$ so that in particular there exists $g \in \mathrm{L}^p_\sigma(\Omega)$ such that

$$\int \langle \nabla u(t), \nabla \psi \rangle \, \mathrm{d}x = \int_\Omega \langle g, \psi \rangle \, \mathrm{d}x \qquad (\psi \in \mathrm{C}^\infty_{c,\sigma}(\Omega)).$$

In this situation, inner regularity, see Galdi [34, Thm. IV.4.1], implies that $u(t) \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{C}^3)$. Now, by Sobolev's embedding theorem, $u(t) \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, so that Hölder's inequality implies that $\langle u(t), \nabla \rangle u(t) \in \mathrm{L}^p_{\mathrm{loc}}(\Omega) \subset \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.

(3) If $\pi(t) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for almost every $t > 0$, one directly sees by an integration by parts, that for all $\varphi \in \mathrm{C}^\infty_c(\Omega; \mathbb{C}^3)$ and almost every $t > 0$

$$\int_\Omega \langle \partial_t u(t) - \Delta u(t) + \langle u(t), \nabla \rangle u(t) + \nabla \pi(t), \varphi \rangle \, \mathrm{d}x = \int_\Omega \langle f(t), \varphi \rangle \, \mathrm{d}x$$

holds. The fundamental lemma of variational calculus then shows that for almost every $t > 0$

$$\partial_t u(t) - \Delta u(t) + \langle u(t), \nabla \rangle u(t) + \nabla \pi(t) = f(t),$$

where the equality has to be understood in $\mathrm{L}^p_{\mathrm{loc}}(\Omega; \mathbb{C}^3)$, holds. We conclude that weak (p, q)-solutions with $\pi(t) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ are in fact strong solutions.

The main result of the second section reads as follows.

Theorem 6.1.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\varepsilon > 0$ be as in Convention 5.2.25. Let further $2 \leq p < 3 + \varepsilon$ and if $p = 2$, let $2 \leq q < \infty$, and if $p > 2$, let*

$$\frac{3 - \frac{p-2}{p} \frac{3+\varepsilon}{1+\varepsilon}}{\frac{3}{2} - \frac{3(p-2)}{2p} \frac{3+\varepsilon}{1+\varepsilon}} < q < \infty.$$

Then there exists $C > 0$ such that for all $a \in (\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q,q}$ and $f \in \mathrm{L}^q(0,\infty; \mathrm{L}^p_\sigma(\Omega))$ with

$$\|a\|_{(\mathrm{L}^p_\sigma(\Omega),\mathcal{D}(A_p))_{1-1/q,q}} + \|f\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))} < C,$$

there exists a weak (p,q)-solution (u,π) with $\pi \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$.

The other solution concept that we consider, is the one of mild solutions. Here, we will assume that the right-hand side f of (NSE) is equal to zero. Mild solutions were already introduced in the beginning of Section 2.2 for inhomogeneous linear abstract Cauchy problems. Note, that for solenoidal vector fields u the identity

$$\langle u, \nabla \rangle u = \sum_{i=1}^d \partial_i (u_i u) - \sum_{i=1}^d \partial_i u_i u = \sum_{i=1}^d \partial_i (u_i u) = \operatorname{div}(u \otimes u),$$

where the tensor product $u \otimes u$ denotes the matrix $(u_i u_j)_{i,j=1}^d$, holds. By virtue of the variation of constants formula (2.2), the following definition is reasonable and in fact the standard definition of mild solutions, see, e.g., Kato and Fujita [60], Kato [59], Deuring and von Wahl [22], and Mitrea and Monniaux [75].

Definition 6.1.4. Let $3 \leq p < 3 + \varepsilon$, with $\varepsilon > 0$ from Convention 5.2.25, and $T_0 \in (0, \infty]$. Given $a \in \mathrm{L}^p_\sigma(\Omega)$, a function $u : [0, T_0) \to \mathrm{L}^p_\sigma(\Omega)$ is called a *mild solution to* (NSE) if for every $q \in (p, 3 + \varepsilon)$ and every $t \in (0, T_0)$ the evaluation $u(t)$ is in $\mathrm{L}^q_\sigma(\Omega)$,

$$(0,t) \ni s \mapsto \mathrm{e}^{-(t-s)A_{q/2}} \mathbb{P}_{q/2} \operatorname{div}(u(s) \otimes u(s)) \in \mathrm{L}^1(0,t; \mathrm{L}^p_\sigma(\Omega)),$$

and if

$$u(t) = \mathrm{e}^{-tA_p} a - \int_0^t \mathrm{e}^{-(t-s)A_{q/2}} \mathbb{P}_{q/2} \operatorname{div}(u(s) \otimes u(s)) \, \mathrm{d}s.$$

Remark 6.1.5. In Definition 6.1.4, we had to assume additionally that $u(t) \in \mathrm{L}^q_\sigma(\Omega)$ for all $q \in (p, 3+\varepsilon)$. This has a technical reason, since for $p = 3$, we do not know whether the operator $\mathrm{e}^{-tA_{3/2}} \mathbb{P}_{3/2} \operatorname{div}$ extends to an operator on all of $\mathrm{L}^{3/2}_\sigma(\Omega)$ into some other function space, Theorem 5.2.22.

The main result of the third section reads as follows.

Theorem 6.1.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\varepsilon > 0$ be as in Convention 5.2.25, and $3 \leq r < \min\{3+\varepsilon, 4\}$. For $a \in \mathrm{L}^r_\sigma(\Omega)$ the following statements are valid.*

(1) There exists a number $T_0 > 0$ and a mild solution u to (NSE) *on $[0, T_0)$ that satisfies for all p with $0 \leq \frac{3}{2}(\frac{1}{r} - \frac{1}{p})$ and $p < \min\{3+\varepsilon, 4\}$*

$$t \mapsto t^{\frac{3}{2}(\frac{1}{r}-\frac{1}{p})} u(t) \in \mathrm{BC}([0, T_0); \mathrm{L}^p_\sigma(\Omega)).$$

Moreover, if $0 < \frac{3}{2}(\frac{1}{r} - \frac{1}{p})$ and $p < \min\{3+\varepsilon, 4\}$, then

$$t^{\frac{3}{2}(\frac{1}{r}-\frac{1}{p})} \|u(t)\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \to 0 \quad \text{as} \quad t \searrow 0$$

and

$$\|u(t) - a\|_{\mathrm{L}^r(\Omega;\mathbb{C}^3)} \to 0 \quad \text{as} \quad t \searrow 0.$$

(2) If $r > 3$, there exists a constant $C > 0$, depending only on r, p, and the constants in the estimates in Theorem 5.2.22 (1) and (3), such that

$$T_0 \geq C \|a\|_{\mathrm{L}^r(\Omega;\mathbb{C}^3)}^{-\frac{4r}{r-3}}.$$

(3) For all $3 \leq p < \min\{3+\varepsilon, 4\}$ there are positive constants $C, \mathcal{C} > 0$, depending only on p and the constants in the estimates in Theorem 5.2.22 (1) and (3), such that if $\|a\|_{\mathrm{L}^3(\Omega;\mathbb{C}^3)} < \mathcal{C}$, the solution of (1) is global, i.e., $T_0 = \infty$. Moreover, it satisfies the estimate

$$\|u(t)\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \leq C t^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{p})} \qquad (0 < t < \infty).$$

Remark 6.1.7. In [41, Thm. 1], Giga states further properties (like uniqueness for special choices of Lebesgue exponents and blow-up rates on the maximal interval of existence) of mild solutions that solely have to satisfy the conclusions of Theorem 6.1.6. It follows that these properties do also hold for the mild solutions constructed in this work. The reader may consult [41] for further reading.

We would like to mention, that the mild solutions for $r = 3$ lie in the space $\mathrm{L}^\infty(0, T_0; \mathrm{L}^3(\Omega; \mathbb{C}^3))$ (or $\mathrm{L}^\infty(0, \infty; \mathrm{L}^3(\Omega; \mathbb{C}^3))$ for small initial data). As was already mentioned in the introduction of this thesis, this is a space that is critical for the Navier-Stokes equations in the sense that this norm is invariant under the natural scaling of solutions to the Navier-Stokes equations. Besides Theorem 6.1.6, also Theorem 6.1.3 produces solutions in this critical space whenever q is large enough. This can be seen by virtue of the embedding proven in Proposition 6.2.1 below.

Furthermore, we would also like to recall, that solutions to the Navier-Stokes equations satisfy the *Serrin condition* if

$$u \in \mathrm{L}^q(0, \infty; \mathrm{L}^p(\Omega; \mathbb{C}^3)), \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1, \quad p \in (3, \infty), \quad q \in (2, \infty).$$

Rearranging yields that

$$q = \frac{2p}{p-3},$$

so that in view of the condition on imposed on q in Theorem 6.1.3, we see that this theorem produces solutions that fulfill the Serrin condition whenever p is close to 3.

6.2 The approach via maximal L^q-regularity

This approach finds its foundations in the work of Solonnikov [91] and was subsequently raised into the abstract machinery of $\mathcal{R}$-boundedness presented in Section 2.2. For a modern approach to maximal L^q-regularity of the Stokes operator on bounded and smooth domains, the reader may consult the work of the authors Geissert, Hess, Hieber, Schwarz, and Stavrakidis [35]. The argument to obtain solutions to the Navier-Stokes equations is straightforward. Firstly, project the Navier-Stokes equations onto $\mathrm{L}^p_\sigma(\Omega)$, to obtain a projected equation

$$\begin{cases} \partial_t u + A_p u = \mathbb{P}_p f - \mathbb{P}_p \langle u, \nabla \rangle u \\ \qquad u(0) = a. \end{cases}$$

Secondly, replace the term $\mathbb{P}_p \langle u, \nabla \rangle u$ on the right-hand side by $\mathbb{P}_p \langle v, \nabla \rangle v$, with some function v in a proper function space. For f and a fixed, consider then the solution operator $\mathcal{T}_{(f,a)}$, which maps v onto u. In order to solve (NSE), the task is then to find a fixed point of $\mathcal{T}_{(f,a)}$. This is the point where the maximal L^q-regularity and the smallness of the data enter the game. In the following, we will perform this argument rigorously.

Let $\mathbb{E}$ be the space defined as

$$\mathbb{E} := \mathrm{L}^q(0,\infty;\mathcal{D}(A_p)) \cap \mathrm{W}^{1,q}(0,\infty;\mathrm{L}^p_\sigma(\Omega)).$$

The following proposition establishes some embedding results for $\mathbb{E}$. These will be important for a control on the nonlinearity of the Navier-Stokes equations, which is proven in Lemma 6.2.2.

Proposition 6.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\varepsilon > 0$ as in Convention 5.2.25, and $2 \leq p < 3 + \varepsilon$. Then, for*

$$\begin{cases} 1 \leq s < \frac{3}{2} - \frac{p-2}{2p}\frac{3+\varepsilon}{1+\varepsilon}, & \text{if } p > 2 \\ 1 \leq s \leq \frac{3}{2}, & \text{if } p = 2 \end{cases}$$

the continuous embedding

$$\mathbb{E} \subset \begin{cases} \mathrm{L}^{\frac{sq}{s-sq+q}}(0,\infty;\mathrm{W}^{1,p}(\Omega;\mathbb{C}^3)), & \text{if } 1 < q < \frac{s}{s-1} \\ \mathrm{L}^\infty(0,\infty;\mathrm{W}^{1,p}(\Omega;\mathbb{C}^3)), & \text{if } \frac{s}{s-1} \leq q < \infty \end{cases}$$

holds.

Proof. The proposition readily follows by combining the embedding results of $\mathcal{D}(A_p)$, see Theorems 5.2.6 and 5.2.9, with Corollary 2.4.4. $\square$

Lemma 6.2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\varepsilon > 0$ be as in Convention 5.2.25, and $2 \leq p < 3 + \varepsilon$. In the case $p = 2$, let $2 \leq q < \infty$, and in the case $p > 2$, let*

$$\frac{3 - \frac{p-2}{p}\frac{3+\varepsilon}{1+\varepsilon}}{\frac{3}{2} - \frac{3(p-2)}{2p}\frac{3+\varepsilon}{1+\varepsilon}} < q < \infty.$$

Then, there exists $C > 0$ such that for all $v, w \in \mathbb{E}$

$$\|\langle v, \nabla\rangle w\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3))} \leq C\|v\|_{\mathbb{E}}\|w\|_{\mathbb{E}}.$$

Proof. Let $v, w \in \mathbb{E}$. Since $\mathcal{D}(A_p) \subset \mathcal{D}(A_2)$ since $p \geq 2$ we find that $v(t) \in \mathcal{D}(A_2)$ for almost every $t > 0$. Thus, by Brown and Shen [13, Thm. 3.1] there exists a constant $C > 0$, independent of v, such that

$$\|v(t)\|_{\mathrm{L}^\infty(\Omega;\mathbb{C}^3)} \leq C \|\nabla v(t)\|^{\frac{1}{2}}_{\mathrm{L}^2(\Omega;\mathbb{C}^9)} \|A_2 v(t)\|^{\frac{1}{2}}_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} \qquad (\text{a.e. } t > 0).$$

Thus, there exists a constant $C > 0$ such that

$$\begin{aligned}&\int_0^\infty \|\langle v(t), \nabla\rangle w(t)\|^q_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \\ &\quad\leq C \int_0^\infty \|\nabla v(t)\|^{\frac{q}{2}}_{\mathrm{L}^2(\Omega;\mathbb{C}^9)} \|A_2 v(t)\|^{\frac{q}{2}}_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} \|\nabla w(t)\|^q_{\mathrm{L}^p(\Omega;\mathbb{C}^9)} \,\mathrm{d}t.\end{aligned}$$

Applying Hölder's inequality in space and time shows with a different constant C

$$\begin{aligned}&\leq C \bigg(\int_0^\infty \|\nabla v(t)\|^q_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \|\nabla w(t)\|^{2q}_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \bigg)^{\frac{1}{2}} \cdot \\ &\qquad \cdot \bigg(\int_0^\infty \|A_p v(t)\|^q_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \bigg)^{\frac{1}{2}}.\end{aligned}$$

Another application of Hölder's inequality in time shows

$$\begin{aligned}&\leq C \bigg(\int_0^\infty \|\nabla v(t)\|^{3q}_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \bigg)^{\frac{1}{6}} \bigg(\int_0^\infty \|\nabla w(t)\|^{3q}_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \bigg)^{\frac{1}{3}} \\ &\qquad \cdot \bigg(\int_0^\infty \|A_p v(t)\|^q_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \,\mathrm{d}t \bigg)^{\frac{1}{2}}.\end{aligned}$$

Finally, we would like to appeal to Proposition 6.2.1. For this purpose, we have to ensure that there exists a number s subject to the premises given in the very same proposition, such that

$$1 < q < \frac{s}{s-1} \quad \text{and} \quad 3q = \frac{sq}{s - sq + q}.$$

To do so, define

$$s := s(q) := \frac{3q}{3q-2}.$$

Calculating the derivative of $s(q)$ shows that s is strictly monotone decreasing, so that

$$s > \lim_{x\to\infty} s(x) = 1.$$

Recalling the conditions imposed on q implies that if $p = 2$, we have

$$s \leq s(2) = \frac{3}{2}.$$

If $2 < p < 3 + \varepsilon$, we find

$$\begin{aligned} s < s\bigg(\frac{3 - \frac{p-2}{p}\frac{3+\varepsilon}{1+\varepsilon}}{\frac{3}{2} - \frac{3(p-2)}{2p}\frac{3+\varepsilon}{1+\varepsilon}}\bigg) &= \frac{9 - \frac{3(p-2)}{p}\frac{3+\varepsilon}{1+\varepsilon}}{9 - \frac{3(p-2)}{p}\frac{3+\varepsilon}{1+\varepsilon} - 2(\frac{3}{2} - \frac{3(p-2)}{2p}\frac{3+\varepsilon}{1+\varepsilon})} \\ &= \frac{9 - \frac{3(p-2)}{p}\frac{3+\varepsilon}{1+\varepsilon}}{6} \\ &= \frac{3}{2} - \frac{p-2}{2p}\frac{3+\varepsilon}{1+\varepsilon}. \end{aligned}$$

Next, one directly calculates

$$\frac{s}{s-1} = \frac{3q}{3q - 3q + 2} > q$$

and

$$\frac{sq}{s - sq + q} = \frac{3q^2}{3q - 3q^2 + q(3q-2)} = 3q.$$

Now, we can use Proposition 6.2.1 to estimate

$$\bigg(\int_0^\infty \|\nabla v(t)\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)}^{3q} \,\mathrm{d}t\bigg)^{\frac{1}{6}} \bigg(\int_0^\infty \|\nabla w(t)\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)}^{3q} \,\mathrm{d}t\bigg)^{\frac{1}{3}} \leq C\|v\|_{\mathbb{E}}^{q/2}\|w\|_{\mathbb{E}}^{q}.$$

This concludes the proof. $\square$

Let us consider the Navier-Stokes equations (NSE). Applying (informally) the Helmholtz projection to the system yields that u must be in $\mathcal{D}(A_p)$ and must satisfy

$$(\mathbb{P}\text{NSE}) \qquad \begin{cases} \partial_t u + A_p u = \mathbb{P}_p f - \mathbb{P}_p\langle u, \nabla\rangle u \\ \qquad\qquad u(0) = a. \end{cases}$$

Next, replace u by $v \in \mathbb{E}$ in the nonlinearity and suppose that $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3))$ with p and q as in the previous lemma. Then, by virtue of Lemma 6.2.2 the right-hand side of this equation lies in $\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))$. Given an initial condition $a \in (\mathrm{L}^p_\sigma(\Omega),\mathcal{D}(A_p))_{1-1/q,q}$, we deduce by the maximal L^q-regularity of the Stokes operator, see Theorem 5.2.24, that for every $v \in \mathbb{E}$ there must be a unique solution u to

$$\begin{cases} \partial_t u + A_p u = \mathbb{P}_p f - \mathbb{P}_p \langle v, \nabla\rangle v \\ \qquad u(0) = a. \end{cases}$$

For fixed $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3))$ and $a \in (\mathrm{L}^p_\sigma(\Omega),\mathcal{D}(A_p))_{1-1/q,q}$, define the operator $\mathcal{T}_{(f,a)} : \mathbb{E} \to \mathbb{E}$, that maps v to u. It is clear that u solves ($\mathbb{P}$NSE) if and only if u is a fixed point of $\mathcal{T}_{(f,a)}$. The following proposition reveals a connection to weak (p,q)-solutions.

Proposition 6.2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\varepsilon > 0$ be as in Convention 5.2.25, and $2 \leq p < 3+\varepsilon$. In the case $p = 2$, let $2 \leq q < \infty$, and in the case $p > 2$, let*

$$\frac{3 - \frac{p-2}{p}\frac{3+\varepsilon}{1+\varepsilon}}{\frac{3}{2} - \frac{3(p-2)}{2p}\frac{3+\varepsilon}{1+\varepsilon}} < q < \infty.$$

Let further $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3))$ and $a \in (\mathrm{L}^p_\sigma(\Omega),\mathcal{D}(A_p))_{1-1/q,q}$.

If $u \in \mathbb{E}$ is a fixed point of $\mathcal{T}_{(f,a)}$, then, for almost every $t > 0$, there exists a pressure $\pi(t) \in \mathrm{L}^p(\Omega) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, such that (u,π) are (p,q)-weak solutions to (NSE).

Proof. Let $u \in \mathbb{E}$ be a fixed point of $\mathcal{T}_{(f,a)}$. Note that, since u is a solution to the abstract Cauchy problem above, u is given by the variation of constants formula (2.2). This representation yields that $u(t)$ converges to a in $\mathrm{L}^p_\sigma(\Omega)$. Moreover, since u is a fixed point of $\mathcal{T}_{(f,a)}$, the identity

$$A_p u = \mathbb{P}_p f - \mathbb{P}_p\langle u, \nabla\rangle u - \partial_t u$$

is valid. Since $u(t) \in \mathcal{D}(A_p)$ for almost every $t > 0$, Theorem 5.2.11 implies that for almost every $t > 0$, we find a pressure function $\phi_1(t) \in \mathrm{L}^p(\Omega)$ such that $A_p u(t) = -\Delta u(t) + \nabla \phi_1(t)$ in the sense of distributions. Note that ϕ_1 lies in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ due to inner regularity, see Galdi [34, Thm. IV.4.2].

Moreover, by Lemma 6.2.2, the function $\langle u(t), \nabla\rangle u(t)$ lies in $\mathrm{L}^p(\Omega;\mathbb{C}^3)$ for almost every $t > 0$. Consequently, for these t, $(\mathbb{P}_p - \mathrm{Id})\langle u(t), \nabla\rangle u(t) = \nabla\phi_2(t)$ for some $\phi_2 \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla\phi_2 \in \mathrm{L}^p(\Omega;\mathbb{C}^3)$ by Lemma 5.1.3 and Theorem 5.1.10. Finally, since $f \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3))$, we have $f(t) \in \mathrm{L}^p(\Omega;\mathbb{C}^3)$ for almost every $t > 0$. For these t, there is a function $\phi_3(t) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\nabla\phi_3 \in \mathrm{L}^p(\Omega;\mathbb{C}^3)$ such that $(\mathbb{P}_p - \mathrm{Id})f(t) = \nabla\phi_3(t)$ by Lemma 5.1.3 and Theorem 5.1.10. Note that [34, Rem. II.6.1] implies that both functions, ϕ_2 and ϕ_3, lie in $\mathrm{W}^{1,p}(\Omega)$. Then, for all $t > 0$ where $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ exist and where $\partial_t u(t) \in \mathrm{L}^p_\sigma(\Omega)$, we find for all $\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^3)$

$$
\begin{aligned}
&\int_\Omega \langle \nabla u(t), \nabla\varphi\rangle - \phi_1(t)\overline{\operatorname{div}(\varphi)}\,\mathrm{d}x \\
&= \int_\Omega \langle \mathbb{P}_p f(t), \varphi\rangle - \langle \mathbb{P}_p\langle u(t), \nabla\rangle u(t), \varphi\rangle - \langle \partial_t u(t), \varphi\rangle\,\mathrm{d}x \\
&= \int_\Omega \langle f(t), \varphi\rangle - \langle\langle u(t), \nabla\rangle u(t) - \partial_t u(t), \varphi\rangle - \langle \nabla[\phi_2(t) - \phi_3(t)], \varphi\rangle\,\mathrm{d}x.
\end{aligned}
$$

An integration by parts shows that we can define

$$\pi(t) := \phi_1(t) + \phi_2(t) - \phi_3(t).$$

Thus, (u,π) are weak (p,q)-solutions with $\pi(t) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for almost every $t > 0$. □

The following theorem shows that $\mathcal{T}_{(f,a)}$ has a unique fixed point, provided f and a are small enough. The proof follows the lines of Saal [84, Thm. 1.2].

Theorem 6.2.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\varepsilon > 0$ be as in Convention 5.2.25, and $2 \leq p < 3+\varepsilon$. In the case $p = 2$, let $2 \leq q < \infty$, and in the case $p > 2$, let*

$$\frac{3 - \frac{p-2}{p}\frac{3+\varepsilon}{1+\varepsilon}}{\frac{3}{2} - \frac{3(p-2)}{2p}\frac{3+\varepsilon}{1+\varepsilon}} < q < \infty.$$

Then, there exists a constant $C > 0$, such that for all

$$(f,a) \in \mathrm{L}^q(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^3)) \times (\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q,q} =: \mathbb{F}$$

with

$$\|(f,a)\|_{\mathbb{F}} < C,$$

there exists a unique fixed point $u \in \mathbb{E}$ *of* $\mathcal{T}_{(f,a)}$.

Proof. Let $\tau_0 : \mathbb{E} \to (\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q,q}$ be the trace operator at zero. This operator is well-defined due to the discussion in Subsection 1.2.2. Define

$$\mathbb{F}_\sigma := \mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega)) \times (\mathrm{L}^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q,q}$$

and the space of solutions corresponding to a zero initial condition

$$\mathbb{E}_0 := \tau_0^{-1}(\{0\}).$$

Furthermore, let $L^{-1} : \mathbb{F}_\sigma \to \mathbb{E}$ denote the solution operator to the problem

$$\begin{cases} \partial_t u + A_p u = g \\ \qquad\quad u(0) = a, \end{cases}$$

which exists due to the maximal L^q-regularity of A_p proven in Theorem 5.2.24 and the discussion at the beginning of Section 2.2. That the solution operator indeed maps into $\mathbb{E}$ follows from Proposition 2.2.3 and the invertibility of A_p. Fix $(f,a) \in \mathbb{F}$ and reformulate the operator $\mathcal{T}_{(f,a)}$ by means of L^{-1} as

$$\mathcal{T}_{(f,a)}(v) = L^{-1}(\mathbb{P}_p f - \mathbb{P}_p \langle v, \nabla \rangle v, a).$$

Define $\widetilde{v} := L^{-1}(\mathbb{P}_p f, a)$ and conclude that by the linearity of L^{-1},

$$\mathcal{T}_{(f,a)}(v) = L^{-1}(-\mathbb{P}_p \langle v, \nabla \rangle v, 0) + \widetilde{v}.$$

Next, let $\overline{v} := v - \widetilde{v}$. By rearranging we see that $\mathcal{T}_{(f,a)}$ has a fixed point if and only if

$$\overline{v} = L^{-1}(-\mathbb{P}_p \langle (\overline{v} + \widetilde{v}), \nabla \rangle (\overline{v} + \widetilde{v}), 0).$$

Thus, $\mathcal{T}_{(f,a)}$ has a fixed point if and only if the mapping

$$\mathrm{S}_{(f,a)} : \mathbb{E}_0 \to \mathbb{E}_0, \quad \overline{v} \mapsto L^{-1}(-\mathbb{P}_p\langle(\overline{v}+\widetilde{v}), \nabla\rangle(\overline{v}+\widetilde{v}), 0)$$

has a fixed point. We proceed by showing that the contraction mapping principle can be applied to the restriction of $\mathrm{S}_{(f,a)}$ to a ball $B(0,r) \subset \mathbb{E}_0$ with a suitable radius. Let $r > 0$ to be chosen and $\overline{v} \in B(0,r)$. Then, by virtue of the maximal regularity of A_p, see Theorem 5.2.24, together with the maximal regularity estimate (2.3), we find

$$\|\mathrm{S}_{(f,a)}(\overline{v})\|_{\mathbb{E}_0} \leq C\|\mathbb{P}_p\langle(\overline{v}+\widetilde{v}), \nabla\rangle(\overline{v}+\widetilde{v})\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))}.$$

According to the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega;\mathbb{C}^3)$ as well as Lemma 6.2.2, with a different constant $C > 0$, this is controlled by

$$\leq C\Big\{\|\overline{v}\|^2_{\mathbb{E}_0} + \|\overline{v}\|_{\mathbb{E}_0}\|\widetilde{v}\|_{\mathbb{E}} + \|\widetilde{v}\|^2_{\mathbb{E}}\Big\}.$$

Invoking Proposition 2.2.3 as well as the boundedness of $\mathbb{P}_p$ shows with a different constant $C > 0$

$$\leq C\Big\{\|\overline{v}\|^2_{\mathbb{E}_0} + \|\overline{v}\|_{\mathbb{E}_0}\|(f,a)\|_{\mathbb{F}} + \|(f,a)\|^2_{\mathbb{F}}\Big\}.$$

Using that $\overline{v} \subset B(0,r)$ reveals

$$\leq C\Big\{r^2 + r\|(f,a)\|_{\mathbb{F}} + \|(f,a)\|^2_{\mathbb{F}}\Big\}.$$

Taking $\|(f,a)\|_{\mathbb{F}} < r$, we obtain that

$$3Cr^2 < r \Leftrightarrow r < \frac{1}{3C}.$$

Consequently, with every r satisfying the preceding inequality on the right-hand side and with the smallness condition on (f,a) in $\mathbb{F}$ the calculation above shows that $\mathrm{S}_{(f,a)}$ maps $B(0,r)$ into itself. To show that the restriction of $\mathrm{S}_{(f,a)}$ to a ball (of proper size) is a strict contraction, calculate with the same reasoning as above

$$\begin{aligned}\|\mathrm{S}_{(f,a)}(\overline{v}_1) - \mathrm{S}_{(f,a)}(\overline{v}_2)\|_{\mathbb{E}_0} &\leq C\Big\{\|\mathbb{P}_p\langle(\overline{v}_1+\widetilde{v}), \nabla\rangle(\overline{v}_1-\overline{v}_2)\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))} \\ &\qquad + \|\mathbb{P}_p\langle(\overline{v}_1-\overline{v}_2), \nabla\rangle(\overline{v}_2+\widetilde{v})\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma(\Omega))}\Big\} \\ &\leq C\Big\{\|\overline{v}_1\|_{\mathbb{E}_0} + \|\widetilde{v}\|_{\mathbb{E}} + \|\overline{v}_2\|_{\mathbb{E}_0}\Big\}\|\overline{v}_1-\overline{v}_2\|_{\mathbb{E}_0}.\end{aligned}$$

Finally, by means (2.3), Proposition 2.2.3, and the boundedness of $\mathbb{P}_p$, with a different constant $C > 0$, this is controlled by

$$\leq C\Big\{\|\overline{v}_1\|_{\mathbb{E}_0} + \|(f,a)\|_{\mathbb{F}} + \|\overline{v}_2\|_{\mathbb{E}_0}\Big\}\|\overline{v}_1 - \overline{v}_2\|_{\mathbb{E}_0}.$$

So, we see, that $\mathrm{S}_{(f,a)}$ is a strict contraction, if $r < 1/3C$. Hence, the contraction mapping principle is applicable and proves the existence of a unique fixed point of $\mathrm{S}_{(f,a)}$ in $B(0,r)$. $\square$

Proof of Theorem 6.1.3. The proof is merely a combination of Theorem 6.2.4 with Proposition 6.2.3. $\square$

6.3 Giga's iteration scheme for semilinear parabolic problems

In this section, we will investigate semilinear parabolic problems of the type

$$\text{(SPP)} \qquad \begin{cases} \partial_t u(t) + Au(t) = Fu(t) & (0 < t < T) \\ \qquad\qquad u(0) = a, \end{cases}$$

where Fu represents the nonlinear part of the equation and $-A$ is the generator of an analytic semigroup on a range of certain subspaces of L^p. This problem was studied by several authors and culminated in the work of Giga in [41]. Please consult this paper for the references of other authors. In that work, he required the validity of merely three estimates on the semigroup and the nonlinearity in order to perform an iteration scheme which establishes the solvability of (SPP) in the mild sense. We will review this proof and thereby slightly weaken some of the required estimates. This enables us to apply this iteration scheme to the Navier-Stokes equations on bounded Lipschitz domains in Subsection 6.3.1.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a locally compact Hausdorff space endowed with a σ finite, complete Radon measure whose σ algebra contains the Borel sets. Let further $l, k \in \mathbb{N}$ and $1 < p_- < p^\circ < p_+ < \infty$ be real numbers. For $p_- < p < p_+$, let U^p be a closed subspace of $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l)$, and let P_p be a continuous projector from $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l)$ onto U^p such that the restriction of P_p to $\mathrm{C}_c(\Omega; \mathbb{C}^l)$ is independent of p. In this setting, the following lemma is valid.

Lemma 6.3.1. *In the setting described before this lemma and with* $p < q < p_+$, *we have that* $\mathrm{P}_p = \mathrm{P}_q$ *on* $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l) \cap L^q(\Omega, \mu; \mathbb{C}^l)$ *and that* $\mathrm{U}^p \cap \mathrm{L}^q(\Omega, \mu; \mathbb{C}^l) \subset \mathrm{U}^q$.

Proof. Note that $\mathrm{C}_c(\Omega; \mathbb{C}^l)$ is dense in $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l)$ and $\mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$ by Rudin [83, Thm. 3.14] and that an analysis of this proof reveals that $f \in \mathrm{L}^p(\Omega, \mu; \mathbb{C}^l) \cap \mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$ can be approximated by a sequence $(f_k)_{k \in \mathbb{N}} \subset \mathrm{C}_c(\Omega; \mathbb{C}^l)$, which converges to f with respect to both topologies.

Now, since P_p and P_q coincide on $\mathrm{C}_c(\Omega; \mathbb{C}^l)$, we have $\mathrm{P}_p f_k = \mathrm{P}_q f_k$ for all $k \in \mathbb{N}$. By boundedness of P_p, $(\mathrm{P}_p f_k)_{k \in \mathbb{N}}$ converges to $\mathrm{P}_p f$ in $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l)$. Moreover, along a subsequence, $(\mathrm{P}_p f_{k_j})_{j \in \mathbb{N}}$ converges μ-almost everywhere to $\mathrm{P}_p f$. The boundedness of P_q implies that $(\mathrm{P}_q f_{k_j})_{j \in \mathbb{N}}$ converges to $\mathrm{P}_q f$. Moreover, a subsequence converges μ-almost everywhere to $\mathrm{P}_q f$. It follows that P_p and P_q coincide on $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^l) \cap L^q(\Omega, \mu; \mathbb{C}^l)$.

Finally, if $f \in \mathrm{U}^p \cap \mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$, we derive $f = \mathrm{P}_p f = \mathrm{P}_q f \in \mathrm{U}^q$. This concludes the proof ☐

We assume the following properties of the quantities arising in (SPP). For brevity, we will write $\|f\|_{\mathrm{U}^p}$ instead of $\|f\|_{\mathrm{L}^p(\Omega,\mu;\mathbb{C}^l)}$ if $f \in \mathrm{U}^p$.

Assumption 6.3.2. *(A) For all* $p_- < p < p_+$ *there are operators* $-A_p : \mathcal{D}(A_p) \subset \mathrm{U}^p \to \mathrm{U}^p$ *that generate analytic semigroups of operators* $(\mathrm{e}^{-tA_p})_{t \geq 0}$ *on* U^p, *which are consistent on the* U^p*-scale, i.e.,* $\mathrm{e}^{-tA_{p_1}}$ *coincides with* $\mathrm{e}^{-tA_{p_2}}$ *on* $\mathrm{U}^{p_1} \cap \mathrm{U}^{p_2}$ *for all* $t \geq 0$ *and* $p_- < p_1, p_2 < p_+$. *Moreover, there are constants* $n, m \geq 1$ *such that for every fixed* $0 < T < \infty$ *there exists* $M_T > 0$ *such that for all* $f \in \mathrm{U}^p$ *and* $0 < t < T$ *the estimate*

$$\|\mathrm{e}^{-tA_p} f\|_{\mathrm{U}^q} \leq M_T t^{-\sigma} \|f\|_{\mathrm{U}^p}$$

holds with $\sigma := \frac{n}{m}(\frac{1}{p} - \frac{1}{q})$ *and* $p_- < p \leq q < p_+$.

(N) There exist constants $\alpha > 0$ *and* $0 \leq \gamma < m$ *such that for each* $p_- < p \leq p^\circ$ *the nonlinear term* Fu *can be decomposed into the composition* $Fu = \Gamma G u$ *with* Γ *and* G *having the following properties. The operator* $\Gamma : \mathcal{D}(\Gamma) \subset \mathrm{L}^p(\Omega, \mu; \mathbb{C}^k) \to \mathrm{U}^p$ *is linear and densely defined. Furthermore, for each* $t > 0$ *and* $p^\circ \leq q < p_+$ *the operator* $\mathrm{e}^{-tA_p}\Gamma$ *extends to a bounded operator from* $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^k)$

into $\mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$*, such that for each given* $0 < T < \infty$ *there exists a constant* $N_{1,T} > 0$ *such that for all* $f \in \mathrm{L}^p(\Omega, \mu; \mathbb{C}^k)$ *and all* $0 < t < T$

$$\|\mathrm{e}^{-tA_p}\Gamma f\|_{\mathrm{U}^q} \leq N_{1,T} t^{-\sigma - \frac{\gamma}{m}} \|f\|_{\mathrm{L}^p(\Omega,\mu;\mathbb{C}^k)},$$

where m *and* σ *are the same numbers as in* (A)*.*

The operator G *is supposed to reflect the nonlinear part and satisfies* $G0 = 0$*. Moreover, there exists a constant* N_2 *such that*

$$\|Gv - Gw\|_{\mathrm{L}^p(\Omega,\mu;\mathbb{C}^k)} \leq N_2 \|v - w\|_{\mathrm{U}^{p(1+\alpha)}} (\|v\|^{\alpha}_{\mathrm{U}^{p(1+\alpha)}} + \|w\|^{\alpha}_{\mathrm{U}^{p(1+\alpha)}})$$

for all $v, w \in \mathrm{U}^{p(1+\alpha)}$*.*

Heuristically, Γ has the role of a differential operator of order γ and Gu behaves like $|u|^{\alpha} u$.

Remark 6.3.3. In comparison to Giga [41], we carried out two changes in these assumptions. First of all, we introduced the numbers p_-, p°, and p_+ and assume that (A) only holds between p_- and p_+. Giga assumed this validity for all $1 < p < \infty$.

Secondly, we replaced the first estimate in (N). In [41], this estimate reads as

$$\|\mathrm{e}^{-tA_p}\Gamma f\|_{\mathrm{U}^p} \leq N_{1,T} t^{-\frac{\gamma}{m}} \|f\|_{\mathrm{L}^p(\Omega,\mu;\mathbb{C}^k)} \tag{6.1}$$

and is assumed for all $1 < p < \infty$. This is the same estimate as ours by taking $p = q$. However, in our situation, this is only possible if $p = p^\circ = q$, i.e., only for one special choice of Lebesgue exponents. Note that Giga's assumptions are stronger than ours, as the estimate

$$\|\mathrm{e}^{-tA_p}\Gamma f\|_{\mathrm{U}^q} \leq N_{1,T} t^{-\sigma - \frac{\gamma}{m}} \|f\|_{\mathrm{L}^p(\Omega,\mu;\mathbb{C}^k)}$$

follows directly by writing $\mathrm{e}^{-tA_p}\Gamma f = \mathrm{e}^{-\frac{t}{2}A_p}\mathrm{e}^{-\frac{t}{2}A_p}\Gamma f$ and by employing the L^p-L^q-estimates from (A) first and then the estimate (6.1).

It is the main insight of our analysis of Giga's proof, that (6.1) is never employed alone, but always in combination with the estimates of (A). This leads to the weaker estimate stated in Assumption 6.3.2 (N).

Fix $p_- < p \leq p^\circ$ and $p^\circ \leq q < p_+$. In (N), we assume that the composition of the semigroup operators e^{-tA_p} with Γ extends to all of $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^k)$ and maps into $\mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$. Now, the question arises whether the semigroup property stays conserved. Note that, by assumption, $\mathcal{D}(\Gamma)$ is dense in $\mathrm{L}^p(\Omega, \mu; \mathbb{C}^k)$ and that Γ maps its domain into U^p. Thus, $\mathrm{e}^{-tA_p}\Gamma f \in \mathrm{U}^p \cap \mathrm{L}^q(\Omega, \mu; \mathbb{C}^l)$ for each $f \in \mathcal{D}(\Gamma)$, so that by Lemma 6.3.1, we conclude that $\mathrm{e}^{-tA_p}\Gamma f \in \mathrm{U}^q$. By the semigroup law and the consistency of the semigroups on the U^p-scale, it follows that

$$\mathrm{e}^{-(s+t)A_p}\Gamma f = \mathrm{e}^{-sA_q}\mathrm{e}^{-tA_p}\Gamma f \qquad (s, t > 0, f \in \mathcal{D}(\Gamma)).$$

Finally, we can use the fact that e^{-sA_q} is bounded on U^q and conclude by density, that the identity above holds for all $f \in \mathrm{U}^p$.

Assumption 6.3.4. *With p_-, p_+, and α as above, assume that*

$$\max\{p_-(1+\alpha), p^\circ\} < \min\{p_+, p^\circ(1+\alpha)\}$$

holds true.

For a in some U^r-space and some $p_- < p \leq p^\circ$, introduce the iteration scheme

$$\begin{aligned} u_0(t) &:= \mathrm{e}^{-tA_r}a, \\ u_{j+1}(t) &:= u_0(t) + Su_j(t) := u_0(t) + \int_0^t \mathrm{e}^{-(t-\tau)A_p} F u_j(\tau)\, \mathrm{d}\tau \qquad (j \in \mathbb{N}_0). \end{aligned}$$

Now, we are in the position to state the theorem.

Theorem 6.3.5. *Under Assumptions 6.3.2 and 6.3.4 the following holds true for every $T > 0$.*

(1) Let p_0, p_0' denote

$$p_0 := \frac{n\alpha}{m - \gamma} \quad \textit{and} \quad p_0' := \max\left\{p_0, p^\circ, \frac{np^\circ(1+\alpha)}{n + p^\circ m}\right\}.$$

Then, $p_0 = p_0'$ holds if $p_0 \geq p^\circ$. Suppose further, that $a \in \mathrm{U}^r$ for a fixed $p_0' < r < \min\{p_+, p^\circ(1+\alpha)\}$ or, if $p_0 > p^\circ$, r could even equal *to p_0. Then there is $0 < T_0 \le T$ and a function u on $[0, T_0)$ that satisfies*

$$\begin{aligned} &t \mapsto t^\sigma u(t) \in \mathrm{BC}([0,T_0);\mathrm{U}^p) && \text{for } r \le p < \min\{p_+, p^\circ(1+\alpha)\}, \\ &t^\sigma \|u(t)\|_{\mathrm{U}^p} \to 0 \quad \text{as} \quad t \to 0 && \text{for } r < p < \min\{p_+, p^\circ(1+\alpha)\}, \end{aligned}$$

whenever $\sigma := \frac{n}{m}(\frac{1}{r} - \frac{1}{p})$ satisfies additionally $0 \le \sigma < \frac{1}{1+\alpha}$. Moreover, for every $\max\{r, p_-(1+\alpha)\} < p < \min\{p_+, p^\circ(1+\alpha)\}$ and $h := \frac{p}{1+\alpha}$, u solves the equation

$$u(t) = \mathrm{e}^{-tA_r}a + \int_0^t \mathrm{e}^{-(t-\tau)A_h}\Gamma G u(\tau)\,\mathrm{d}\tau \qquad (t \in (0,T_0)). \tag{6.2}$$

(2) If $r > p_0'$, the lifespan has the lower bound

$$T_0 \ge C\|a\|_{\mathrm{U}^r}^{-\frac{\alpha}{1-\beta(r)}} \tag{6.3}$$

with $\beta(r) := \frac{1}{m}(\gamma + \frac{n\alpha}{r})$ and $C > 0$ depending only on α, M_T, $N_{1,T}$, N_2, γ, n, m, r, and p.

(3) If $p_0 > p^\circ$ and

$$p_0 \le p < \min\{p_+, p^\circ(1+\alpha)\} \quad \text{with} \quad \frac{n}{m}\Big(\frac{1}{p_0} - \frac{1}{p}\Big) < \frac{1}{1+\alpha},$$

there is a positive constant $\mathcal{C}$ such that if $\|a\|_{\mathrm{U}^{p_0}} < \mathcal{C}$, then T_0 equals T. Furthermore, if M_T and $N_{1,T}$ are uniformly bounded in $0 < T < \infty$, we can take $T = \infty$ and get

$$\|u(t)\|_{\mathrm{U}^p} \le Ct^{-\sigma}, \qquad (0 < t < \infty),$$

with C and $\mathcal{C}$ depending only on α, M_T, $N_{1,T}$, N_2, γ, n, m, r, and p (if $T = \infty$ these constants depend on the uniform bounds of M_T and $N_{1,T}$).

Proof. We start by verifying that $p_0 = p_0'$ holds if $p_0 \ge p^\circ$. By definition of p_0 and p_0', we have to prove that

$$\frac{np^\circ(1+\alpha)}{n + p^\circ m} \le p_0.$$

Calculating the derivative of $p^\circ \mapsto np^\circ(1+\alpha)/(n+p^\circ m)$ shows that this function increases strictly. Thus, due to $p^\circ \le p_0$, we estimate

$$\frac{np^\circ(1+\alpha)}{n+p^\circ m} \le \frac{np_0(1+\alpha)}{n+p_0 m} = \frac{n\alpha(1+\alpha)}{m(1+\alpha)-\gamma} \le \frac{n\alpha(1+\alpha)}{m(1+\alpha)-\gamma(1+\alpha)} = p_0.$$

Next, relying on Assumption 6.3.2 (N), we establish the following estimate. Assume that $p_-(1+\alpha) < p \le p^\circ(1+\alpha)$ and $p^\circ \le s < p_+$. Define $\delta := \frac{n}{m}(\frac{1}{s} - \frac{1}{p})$, $h := p/(1+\alpha)$, and $\beta(p)$ as in (2) of the theorem. Then, for all $0 < t < T$ and all v, $w \in \mathrm{U}^p$ there is a constant $C > 0$ such that

$$\|\mathrm{e}^{-tA_h}(Fv - Fw)\|_{\mathrm{U}^s} \le N_{1,T} N_2 t^{\delta-\beta(p)} \|v-w\|_{\mathrm{U}^p}(\|v\|_{\mathrm{U}^p}^\alpha + \|w\|_{\mathrm{U}^p}^\alpha). \tag{6.4}$$

To establish (6.4), use the first estimate of (N) to derive

$$\begin{aligned}\|\mathrm{e}^{-tA_h}(Fv - Fw)\|_{\mathrm{U}^s} &= \|\mathrm{e}^{-tA_h}\Gamma[Gv - Gw]\|_{\mathrm{U}^s} \\ &\le N_{1,T} t^{\delta-\beta(p)} \|Gv - Gw\|_{\mathrm{L}^h(\Omega,\mu;\mathbb{C}^k)}.\end{aligned}$$

We conclude the proof of (6.4) by applying the second estimate of (N).

Next, we fix a number $0 < T_0 \le T$ and continue by establishing an a priori estimate for the quantity

$$K_j := K_j(T_0) := \sup_{0<t<T_0} t^\sigma \|u_j(t)\|_{\mathrm{U}^p} \qquad (j \ge 0),$$

where σ and p are chosen such that $\sigma = \frac{n}{m}(\frac{1}{r} - \frac{1}{p})$ and

$$\begin{aligned}&\max\Big\{0, \frac{n}{m}\Big(\frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)}\Big)\Big\} < \sigma \qquad \text{and} \\ &\sigma < \min\Big\{\frac{1}{1+\alpha}, \frac{n}{m}\Big(\frac{1}{r} - \frac{1}{\min\{p_+, p^\circ(1+\alpha)\}}\Big)\Big\}.\end{aligned} \tag{6.5}$$

If this choice is possible, this implies

$$p < \min\{p_+, p^\circ(1+\alpha)\}, \quad p > \max\{p_-(1+\alpha), p^\circ\}, \quad \text{and} \quad p > r,$$

so that especially $p > p_0$. Note that due to $\alpha > 0$, $r < \min\{p_+, p^\circ(1+\alpha)\}$, and $\max\{p_-(1+\alpha), p^\circ\} < \min\{p_+, p^\circ(1+\alpha)\}$ (by Assumption 6.3.4) it suffices to prove that

$$\frac{n}{m}\Big(\frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)}\Big) < \frac{1}{1+\alpha}$$

holds, in order to show that the choice of σ in (6.5) is possible. If $r > p_0'$ is satisfied, this follows by

$$p_0' \geq \frac{np^\circ(1+\alpha)}{n+p^\circ m} \qquad \Leftrightarrow \qquad \begin{aligned} \frac{1}{1+\alpha} &\geq \frac{n}{m}\Big(\frac{1}{p_0'} - \frac{1}{p^\circ(1+\alpha)}\Big) \\ &\geq \frac{n}{m}\Big(\frac{1}{p_0'} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)}\Big). \end{aligned}$$

If $p_0 > p^\circ$ and $r = p_0$ the definition of p_0 shows that

$$\frac{1+\alpha}{p_0} = \frac{1}{p_0} + \frac{m}{n} - \frac{\gamma}{n} \leq \frac{1}{p_0} + \frac{m}{n} < \frac{1}{p^\circ} + \frac{m}{n} \leq \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}} + \frac{m}{n}$$

what proves the desired inequality. To establish the a priori bound for K_j recall the iteration scheme defined via

$$u_{j+1} = u_0 + Su_j.$$

An application of the auxiliary estimate (6.4) with $v = u_j, w = 0$, and $s = p$ to the term Su_j yields

$$t^\sigma \|Su_j(t)\|_{\mathrm{U}^p} \leq t^\sigma \int_0^t \frac{N_{1,T}N_2}{(t-\tau)^{\beta(p)}} \|u_j(\tau)\|_{\mathrm{U}^p}^{1+\alpha} \,\mathrm{d}\tau; \tag{6.6}$$

here $\max\{p_-(1+\alpha), p^\circ\} < p < \min\{p_+, p^\circ(1+\alpha)\}$ is used. Since $\beta(r) = \beta(p) + \sigma\alpha$, this gives the iterative estimate

$$K_{j+1} \leq K_0 + N_{1,T}N_2 B K_j^{1+\alpha} T_0^{1-\beta(r)}$$

with

$$B = \int_0^1 \frac{1}{(1-\tau)^{\beta(p)}} \frac{1}{\tau^{\sigma(1+\alpha)}} \,\mathrm{d}\tau. \tag{6.7}$$

The properties $\sigma < \frac{1}{1+\alpha}$ and $p > p_0$ together with $\beta(p_0) = 1$ ensure the convergence. For a technical reason, we use a less sharp estimate

$$K_{j+1} \leq K_0 + 2N_{1,T}N_2 B T_0^{1-\beta(r)} K_j^{1+\alpha}.$$

Assume for a moment, that the inequality

$$T_0^{1-\beta(r)} K_0^\alpha < \Big(\frac{\alpha}{1+\alpha}\Big)^\alpha \frac{1}{2(1+\alpha)N_{1,T}N_2B} \tag{6.8}$$

is valid. Define $K := K(T_0) := \frac{1+\alpha}{\alpha} K_0(T_0)$. Then, by an elementary calculation, we find that

$$K_j < K \qquad (j \geq 0), \tag{6.9}$$

$$2N_1N_2BT_0^{1-\beta(r)}K^\alpha < \frac{1}{1+\alpha}, \tag{6.10}$$

and

$$K \to 0 \quad \text{as} \quad K_0 \to 0. \tag{6.11}$$

We thus have an a priori bound for K_j under the condition (6.8).

In the following, we deduce conditions for T_0 and a that guarantee (6.8). First, we start by proving that

$$t^\sigma \|\mathrm{e}^{-tA_r} a\|_{\mathrm{U}^p} \to 0 \quad \text{as} \quad t \to 0; \tag{6.12}$$

here the property $p > r$ is essential. By density of $\mathrm{C}_c(\Omega; \mathbb{C}^l)$ in $\mathrm{L}^r(\Omega, \mu; \mathbb{C}^l)$, there is a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathrm{C}_c(\Omega; \mathbb{C}^l)$ with $a_i \to a$ in $\mathrm{L}^r(\Omega, \mu; \mathbb{C}^l)$. Moreover, it was assumed that the projection P_q is independent of q on $\mathrm{C}_c(\Omega; \mathbb{C}^l)$ so that $\mathrm{P}_q a_i \in \mathrm{U}^q$ for all $p_- < q < p_+$. Applying the estimate from (A), we deduce

$$\begin{aligned} t^\sigma \|\mathrm{e}^{-tA_r} a\|_{\mathrm{U}^p} &\leq t^\sigma \|\mathrm{e}^{-tA_r}(a - \mathrm{P}_r a_i)\|_{\mathrm{U}^p} + t^\sigma \|\mathrm{e}^{-tA_r} \mathrm{P}_r a_i\|_{\mathrm{U}^p} \\ &\leq M_T \|\mathrm{P}_r a - \mathrm{P}_r a_i\|_{\mathrm{U}^r} + t^{\frac{n}{2m}(\frac{1}{r} - \frac{1}{p})} \|\mathrm{P}_r a_i\|_{\mathrm{U}^{\frac{2rp}{r+p}}}. \end{aligned}$$

Since $p > r$, this reveals (6.12). Particularly, this proves that

$$K_0(T_0) \to 0 \quad \text{as} \quad T_0 \to 0. \tag{6.13}$$

Next, we verify that the assumptions of the theorem imply the validity of (6.8).

If $r > p_0'$ (consequently $\beta(r) < 1$), the condition $T_0 < C\|a\|_{\mathrm{U}^r}^{-\frac{\alpha}{1-\beta(r)}}$ ensures the validity of (6.8) since (A) implies that $K_0/\|a\|_{\mathrm{U}^r}$ is bounded by M_T. Indeed,

$$T_0^{1-\beta(r)}K_0^\alpha \le C\Big(\frac{K_0}{\|a\|_{\mathrm{U}^r}}\Big)^\alpha \le CM_T^\alpha\Big(\frac{\alpha}{1+\alpha}\Big)^\alpha\frac{1}{2(1+\alpha)N_{1,T}N_2B}$$

if C is given by

$$C = \Big[\Big(\frac{\alpha}{1+\alpha}\Big)^\alpha\frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^\alpha}\Big]^{\frac{1}{1-\beta(r)}}.$$

Thus, the estimate (6.3) on the lifespan follows.

If $p_0 > p^\circ$ and $r = p_0$, the convergence (6.13) shows that (6.8) is valid, whenever T_0 is small. Moreover, since $\beta(p_0) = 1$, (6.8) does not include T_0 explicitly, so that (6.8) is satisfied with $T_0 = T$ whenever

$$\|a\|_{\mathrm{U}^{p_0}} < \frac{\alpha}{1+\alpha}\Big(\frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^\alpha}\Big)^{\frac{1}{\alpha}}.$$

Furthermore, if M_T and $N_{1,T}$ are uniformly bounded in $0 < T < \infty$, K is finite for $T_0 = \infty$ and (6.8) is satisfied for small $\|a\|_{\mathrm{U}^{p_0}}$ as well. We thus see that (6.8) holds under all assumptions of the theorem.

So far, we proved a priori estimates (6.9) and (6.10), and also the convergence property (6.11). To see the existence of the function u stated in the theorem, it remains to prove the convergence of $(u_j)_{j\in\mathbb{N}}$ and to check that the limit function is the desired function u. First, we concentrate on proving that $(t^\sigma u_j)_{j\in\mathbb{N}}$ converges in $\mathrm{BC}([0,T_0);\mathrm{U}^p)$ provided σ and p satisfy (6.5). For this purpose, we first show that each $t^\sigma u_j$ is contained in this space. By the a priori bound (6.9), we immediately deduce that $t^\sigma u_j \in \mathrm{L}^\infty(0,T_0;\mathrm{U}^p)$.

We continue by proving continuity. Here, we will spend some effort in order to prove continuity with respect to a broader spectrum of L^s-topologies. This will be helpful for the conclusion of the proof. Fix $\max\{r, \frac{p}{1+\alpha}\} \le s \le p, t_0 \in (0,T_0)$, and let $t > 0$ be small enough. Then, the term u_0 is handled easily by invoking (A). Indeed,

$$\|\mathrm{e}^{-(t_0+t)A_r}a - \mathrm{e}^{-t_0A_r}a\|_{\mathrm{U}^s} \le M_Tt_0^{-\frac{n}{m}(\frac{1}{r}-\frac{1}{s})}\|\mathrm{e}^{-tA_r}a - a\|_{\mathrm{U}^r} \to 0 \quad \text{as} \quad t \to 0$$

by the strong continuity of $(\mathrm{e}^{-tA_r})_{t\geq 0}$ on U^r. If t is replaced by $-t$, the same inequality holds true with $(t_0-t)^{-\frac{n}{m}(\frac{1}{r}-\frac{1}{s})}$ instead of $t_0^{-\frac{n}{m}(\frac{1}{r}-\frac{1}{s})}$.

For the rest of this proof note that due to $p \geq s \geq r \geq p_0' \geq p^\circ$, and (6.5) the numbers s and p satisfy the premise of (6.4). Recall that h was defined to be $p/(1+\alpha)$. Now, the continuity of the function $t \mapsto u_{j+1}(t)$ in the L^s-topology can be concluded via induction. Indeed, taking the continuity of $t \mapsto u_j(t)$ for granted, we have to prove that $t \mapsto Su_j(t)$ is continuous. For this purpose, we estimate for fixed $0 < \tau_0 < t < T_0$ and $0 < \tau < \frac{1}{2}(t-\tau_0) =: t'$ by the triangle inequality

$$\begin{aligned}
&\|\mathrm{e}^{-(t-\tau_0-\tau)A_h} Fu_j(\tau_0+\tau) - \mathrm{e}^{-(t-\tau_0)A_h} Fu_j(\tau_0)\|_{\mathrm{U}^s} \\
&\qquad\leq \|\mathrm{e}^{-(t-\tau_0-\tau)A_h}\Gamma[Gu_j(\tau_0+\tau) - Gu_j(\tau_0)]\|_{\mathrm{U}^s} \\
&\qquad\quad + \|\mathrm{e}^{-(t-\tau_0-\tau-t')A_s}[\mathrm{Id} - \mathrm{e}^{-\tau A_s}]\mathrm{e}^{-t'A_h}\Gamma Gu_j(\tau_0)\|_{\mathrm{U}^s}.
\end{aligned}$$

The first term on the right-hand side is convergent by virtue of (6.4) and the induction hypothesis to zero as $\tau \to 0$. For the second term on the right-hand side, note that by the a priori bounds $u_j(\tau)$ lies in $\mathrm{L}^p(\Omega,\mu;\mathbb{C}^l)$, so that (N) implies that $\mathrm{e}^{-t'A_h}\Gamma Gu_j(\tau_0)$ lies in U^s. The convergence is then derived by the strong continuity of the semigroup in U^s.

As essentially the same calculation works out if one replaces τ by $-\tau$ (here one can skip the usage of the t'), we conclude that the integrand defining Su_j is continuous with respect to the L^s-topology. Thus, for $t_0 \in (0,T_0)$ and $0 < t < T_0 - t_0$ small enough, we find by the triangle inequality

$$\begin{aligned}
&\Big\| \int_0^{t_0+t} \mathrm{e}^{-(t_0+t-\tau)A_h} Fu_j(\tau)\,\mathrm{d}\tau - \int_0^{t_0} \mathrm{e}^{-(t_0-\tau)A_h} Fu_j(\tau)\,\mathrm{d}\tau \Big\|_{\mathrm{L}^s(\Omega,\mu;\mathbb{C}^l)} \\
&\qquad\leq \int_{t_0}^{t_0+t} \|\mathrm{e}^{-(t_0+t-\tau)A_h} Fu_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau \\
&\qquad\quad + \int_0^{t_0} \|[\mathrm{e}^{-tA_s} - \mathrm{Id}]\mathrm{e}^{-(t_0-\tau)A_h} Fu_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau.
\end{aligned}$$

Apply (6.4) with $v = u_j$, and $w = 0$ to the first term to obtain

$$\begin{aligned}
&\leq N_{1,T}N_2K_j \int_{t_0}^{t_0+t} \frac{1}{(t_0+t-\tau)^{\beta(p)-\delta}} \frac{1}{\tau^{\sigma(1+\alpha)}}\,\mathrm{d}\tau \\
&\quad + \int_0^{t_0} \|[\mathrm{e}^{-tA_s} - \mathrm{Id}]\mathrm{e}^{-(t_0-\tau)A_h} Fu_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau.
\end{aligned}$$

The second integral converges to zero by means of the dominated convergence theorem (which is applicable due to the uniform boundedness of the semigroup on U^s by (A), and due to (6.4)) and the strong continuity of the semigroup on U^s. Thus, it remains to investigate the first integral. By performing the substitution $x = \tau/(t_0+t)$ we find that this coincides with

$$(t_0+t)^{1+\delta-\sigma(1+\alpha)-\beta(p)} \int_{\frac{t_0}{t_0+t}}^{1} \frac{1}{(1-x)^{\beta(p)-\delta}} \frac{1}{x^{\sigma(1+\alpha)}} \,\mathrm{d}x.$$

Due to the integrability of the integrand on $[0,1]$, this term tends to zero as $t \to 0$. For the left-handed continuity, calculate as above

$$\begin{aligned}\Big\| \int_0^{t_0-t} \mathrm{e}^{-(t_0-t-\tau)A_h} F u_j(\tau)\,\mathrm{d}\tau - \int_0^{t_0} \mathrm{e}^{-(t_0-\tau)A_h} F u_j(\tau)\,\mathrm{d}\tau \Big\|_{\mathrm{L}^s(\Omega,\mu;\mathbb{C}^l)} \\ \leq \int_{t_0-t}^{t_0} \|\mathrm{e}^{-(t_0-\tau)A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau \\ + \int_0^{t_0-t} \|[\mathrm{Id} - \mathrm{e}^{-tA_s}]\mathrm{e}^{-(t_0-t-\tau)A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau.\end{aligned}$$

The first integral can be treated similarly as the first integral in the case of the right-handed continuity. For the second integral choose $0 < 2t' < t_0$, let $t < t'$, and calculate

$$\begin{aligned}\int_0^{t_0-2t'} \|[\mathrm{Id} - \mathrm{e}^{-tA_s}]\mathrm{e}^{-(t_0-t-\tau)A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau \\ = \int_0^{t_0-2t'} \|\mathrm{e}^{-(t_0-t-\tau-t')A_s}[\mathrm{Id} - \mathrm{e}^{-tA_s}]\mathrm{e}^{-t'A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau.\end{aligned}$$

Due to (A), we get

$$\leq M_T \int_0^{t_0-2t'} \|[\mathrm{Id} - \mathrm{e}^{-tA_s}]\mathrm{e}^{-t'A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau$$

this converges to zero by the dominated convergence theorem (which is applicable due to the uniform boundedness of the semigroup on U^s by (A), and due to (6.4)) and the strong continuity of the semigroup. It remains to control

$$\begin{aligned}\int_{t_0-2t'}^{t_0-t} \|[\mathrm{Id} - \mathrm{e}^{-tA_s}]\mathrm{e}^{-(t_0-t-\tau)A_h} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau \\ \leq 2M_T \int_{t_0-2t'}^{t_0-t} \|\mathrm{e}^{-(t_0-t-\tau)A_s} F u_j(\tau)\|_{\mathrm{U}^s}\,\mathrm{d}\tau.\end{aligned}$$

By means of (6.4), this can be estimated by

$$\leq 2MN_{1,T}N_2K^{1+\alpha}\int_{t_0-2t'}^{t_0-t}\frac{1}{(t_0-t-\tau)^{\beta(p)-\delta}}\frac{1}{\tau^{\sigma(1+\alpha)}}\,\mathrm{d}\tau.$$

Appealing to the substitution $\tau = x(t_0-t)$, the latter integral coincides with

$$(t_0-t)^{1+\delta-\beta(p)-\sigma(1+\alpha)}\int_{\frac{t_0-2t'}{t_0-t}}^{1}\frac{1}{(1-x)^{\beta(p)-\delta}}\frac{1}{x^{\sigma(1+\alpha)}}\,\mathrm{d}x,$$

which converges to zero as $t'\to 0$. This concludes the proof of the left-handed continuity.

Focussing for a moment on the case $s=p$, we conclude that $t^\sigma u_j \in \mathrm{BC}((0,T_0);\mathrm{U}^p)$. Finally, (6.13), (6.11), and (6.9) show that $t^\sigma u_j$ are contained in $\mathrm{BC}([0,T_0);\mathrm{U}^p)$ with

$$\lim_{t\searrow 0} t^\sigma u_j(t) = 0.$$

To show convergence, we consider the successive difference of the u_j's

$$u_{j+1}-u_j = Su_j - Su_{j-1}.$$

Applying (6.4) with $s=p$, $v=u_j$, and $w=u_{j-1}$ together with (6.9) delivers

$$\begin{aligned}
&t^\sigma\|u_{j+1}(t)-u_j(t)\|_{\mathrm{U}^p}\\
&\qquad\leq 2N_{1,T}N_2K^\alpha\int_0^t\frac{1}{(t-\tau)^{\beta(p)}}\frac{1}{\tau^{\sigma(1+\alpha)}}\,\mathrm{d}\tau\sup_{0<\tau<T_0}\tau^\sigma\|u_j(\tau)-u_{j-1}(\tau)\|_{\mathrm{U}^p}\\
&\qquad\leq 2N_{1,T}N_2BT_0^{1-\beta(r)}K^\alpha\sup_{0<\tau<T_0}\tau^\sigma\|u_j(\tau)-u_{j-1}(\tau)\|_{\mathrm{U}^p},
\end{aligned}$$

where B is given in (6.7). Note that the constant $2N_{1,T}N_2BT_0^{1-\beta(r)}K^\alpha$ is strictly less than 1 by (6.10). Consequently, the series

$$\sum_{j=0}^{\infty} t^\sigma[u_{j+1}-u_j]$$

converges absolutely in $\mathrm{BC}([0,T_0);\mathrm{U}^p)$. Considering a partial sum, we see that

$$\sum_{j=0}^{J} t^\sigma[u_{j+1}-u_j] = t^\sigma[u_{J+1}-u_0].$$

Thus, $(t^\sigma u_j)_{j\in\mathbb{N}}$ converges to a function $t^\sigma u$ in $\mathrm{BC}([0,T_0);\mathrm{U}^p)$. An application of (6.4) with $s=p, v=u_j$, and $w=u$ reveals

$$\begin{aligned}
&\|Su - Su_j\|_{\mathrm{U}^p}\\
&\quad\le \int_0^t \|\mathrm{e}^{-(t-\tau)A_h}\Gamma[Gu(\tau)-Gu_j(\tau)]\|_{\mathrm{U}^p}\,\mathrm{d}\tau\\
&\quad\le 2N_{1,T}N_2K^\alpha\int_0^t \frac{1}{(t-\tau)^{\beta(p)}}\frac{1}{\tau^{\sigma(1+\alpha)}}\,\mathrm{d}\tau \sup_{0<\tau<T_0}\tau^\sigma\|u(\tau)-u_j(\tau)\|_{\mathrm{U}^p}
\end{aligned}$$

so that we can pass to the limit in the equation $u_{j+1}=u_0+Su_j$ and deduce that u satisfies (6.2). Furthermore, $t^\sigma u$ takes the value zero at $t=0$ as this is valid for each member of the sequence $(t^\sigma u_j)_{j\in\mathbb{N}}$.

To complete the proof, we have to relax the condition imposed on p in (6.5). Let $\max\{r,\frac{p}{1+\alpha}\}\le s\le p$ and $\sigma' := \frac{n}{m}(\frac{1}{r}-\frac{1}{s})$. By the paragraph dealing with the continuity, we see that $t\mapsto t^{\sigma'}u_j$ is continuous with respect to the L^s-topology. Moreover, applying (6.4) and (6.9) yields

$$\begin{aligned}
t^{\sigma'}\|u_j(t)\|_{\mathrm{U}^s} &\le \sup_{0<t<T_0} t^{\sigma'}\|u_0(t)\|_{\mathrm{U}^s}\\
&\qquad + N_{1,T}N_2t^{\sigma'}K^{1+\alpha}\int_0^t\frac{1}{(t-\tau)^{\beta(p)-\delta}}\frac{1}{\tau^{\sigma(1+\alpha)}}\,\mathrm{d}\tau\\
&= \sup_{0<t<T_0} t^{\sigma'}\|u_0(t)\|_{\mathrm{U}^s} + N_{1,T}N_2CT_0^{1-\beta(r)}K^{1+\alpha},
\end{aligned}\tag{6.14}$$

where

$$C := \int_0^1 \frac{1}{(1-x)^{\beta(p)-\delta}}\frac{1}{x^{\sigma(1+\alpha)}}\,\mathrm{d}x.$$

Note that this integral converges because $p_0<p$ implies $\beta(p)<1$. Consequently, $t\mapsto t^{\sigma'}u_j(t)\in\mathrm{BC}((0,T_0);\mathrm{U}^s)$. Furthermore, as in (6.12) one shows that $\sup_{0<t<T_0}t^{\sigma'}\|u_0(t)\|_s$ converges to zero if $T_0\to 0$ and $s>r$. Combining this with the fact that K converges to zero if K_0 converges to zero by (6.11), (6.13) and (6.14) imply that $t^{\sigma'}u_j$ has a continuation with value zero at $t = 0$. In the case $s = r$, one derives for $0 < t < T_0$ as in (6.14)

$$\|u_j(t) - \mathrm{e}^{-tA_r}a\|_{\mathrm{U}^r} \leq N_{1,T}N_2CT_0^{1-\beta(r)}K^{1+\alpha} \to 0 \quad \text{as} \quad T_0 \to 0.$$

Note that this convergence follows by a combination of (6.11) and (6.13), as in the case $r = p_0$ the number $\beta(r) = 1$. It follows that $u_j \in \mathrm{BC}([0,T_0);\mathrm{U}^r)$ and that each u_j takes the value a at $t = 0$. Finally,

$$\begin{aligned} t^{\sigma'}&\|u_i(t){-}u_j(t)\|_{\mathrm{U}^s} \\ &\leq 2N_{1,T}N_2CT_0^{1-\beta(r)}K^{\alpha} \sup_{0<\tau<T_0} \tau^{\sigma}\|u_{i-1}(\tau) - u_{j-1}(\tau)\|_{\mathrm{U}^p}. \end{aligned}$$

Thus, $(t^{\sigma'}u_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{BC}([0,T_0);\mathrm{U}^s)$. As the proof holds for every p defining a σ which satisfies (6.5), the existence result holds for the stated range of p's. □

6.3.1 Application to the Navier-Stokes equations

We will now apply Theorem 6.3.5 to the Navier-Stokes system (NSE) and thereby prove Theorem 6.1.6.

Proof of Theorem 6.1.6. For this purpose, recall the projected equation ($\mathbb{P}$NSE) with $f = 0$, i.e.,

$$(\mathbb{P}\mathrm{NSE}) \qquad \begin{cases} \partial_t u(t) + A_p u(t) = -\mathbb{P}_p\langle u(t), \nabla\rangle u(t) & (0 < t < \infty) \\ u(0) = a. \end{cases}$$

We define $\mathrm{U}^p := \mathrm{L}^p_\sigma(\Omega)$ and $\mathrm{P}_p := \mathbb{P}_p$. Thereby, fixing the number l as 3. Since $\mathbb{P}_p$ is the restriction of $\mathbb{P}_2$ if $p > 2$ and the extension of $\mathbb{P}_2$ if $p < 2$, it is clear that the action of $\mathbb{P}_p$ under $\mathrm{C}_c(\Omega;\mathbb{C}^3)$ is independent of p. Furthermore, by Theorem 5.1.10, we know that $\mathbb{P}_p$ is a continuous projection on $\mathrm{L}^p(\Omega;\mathbb{C}^3)$, whenever $3/2 - \varepsilon < p < 3 + \varepsilon$.

For the analyticity of the Stokes semigroup, we know by Shen's theorem, Theorem 5.2.19, that $-A_p$ generates a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$ if $3/2 - \varepsilon < p < 3 + \varepsilon$. Moreover, by Theorem 5.2.22 the L^p-L^q-estimates

$$\|\mathrm{e}^{-tA_p}f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^3)} \leq Mt^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^3)} \qquad (f \in \mathrm{L}^p_\sigma(\Omega), t > 0)$$

holds for $3/2-\varepsilon < p \leq q < 3+\varepsilon$. This sets the constants n and m to $n = 3$ and $m = 2$. That the semigroups $(\mathrm{e}^{-tA_p})_{t\geq 0}$ are consistent on the $\mathrm{L}^p_\sigma(\Omega)$-scale follows by Theorem 5.2.22. This shows that Assumption 6.3.2 (A) is valid with the choices above.

In view of (N), we have to write the nonlinearity as

$$-\mathbb{P}_p \langle u, \nabla \rangle u = \Gamma G u,$$

with G incorporating the nonlinear part of ΓG. Recall that for solenoidal vector fields

$$\langle u, \nabla \rangle u = \operatorname{div}(u \otimes u).$$

This suggests to take

$$\Gamma := -\mathbb{P}_p \operatorname{div} \quad \text{and} \quad Gu := u \otimes u.$$

By Hölder's inequality, it is clear that G maps $\mathrm{L}^{2p}(\Omega; \mathbb{C}^3)$ into $\mathrm{L}^p(\Omega; \mathbb{C}^{3\times 3})$ for all $p \geq 1$. Moreover, G satisfies

$$\| Gv - Gw \|_{\mathrm{L}^p(\Omega;\mathbb{C}^{3\times 3})} \leq N_2 \| v - w \|_{\mathrm{L}^{2p}(\Omega;\mathbb{C}^3)} \Big(\| v \|_{\mathrm{L}^{2p}(\Omega;\mathbb{C}^3)} + \| w \|_{\mathrm{L}^{2p}(\Omega;\mathbb{C}^3)} \Big)$$

with an absolute constant $N_2 > 0$. This fixes the choice of α as 1.

The operator Γ is defined a priori on $\mathrm{C}_c^\infty(\Omega; \mathbb{C}^{3\times 3})$ and by virtue of Theorem 5.2.22, the operator $\mathrm{e}^{-tA_p}\mathbb{P}_p \operatorname{div}$ extends to a bounded operator from $\mathrm{L}^p(\Omega; \mathbb{C}^{3\times 3})$ into $\mathrm{L}^q(\Omega; \mathbb{C}^3)$, satisfying for all $t > 0$ and $F \in \mathrm{L}^p(\Omega; \mathbb{C}^{3\times 3})$

$$\| \mathrm{e}^{-tA_p} \mathbb{P}_p \operatorname{div}(F) \|_{\mathrm{L}^q(\Omega;\mathbb{C}^3)} \leq N_1 t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \| F \|_{\mathrm{L}^p(\Omega;\mathbb{C}^{3\times 3})}.$$

Here $3/2 < p \leq 2 \leq q < 3 + \varepsilon$. This choice fixes k as 9 (modulo an identification of $\mathbb{C}^{3\times 3}$ with $\mathbb{C}^9$) and γ as 1.

Thus, we see that the assumptions of Theorem 6.3.5 are satisfied with $p_- := 3/2$, $p^\circ := 2$, and $p_+ := 3 + \varepsilon$. A computation of p_0 delivers

$$p_0 = \frac{n\alpha}{m - \gamma} = \frac{3 \cdot 1}{2 - 1} = 3.$$

Because $p^\circ = 2$, we find $p_0 > p^\circ$, so that we obtain the existence of mild solutions for the Navier-Stokes equations for initial conditions in $\mathrm{L}^p_\sigma(\Omega)$ with $3 \leq p < 3 + \varepsilon$ as described in Theorem 6.3.5. $\square$

CHAPTER 7

Maximal regularity for higher-order elliptic systems in divergence form

The subject of this chapter is a little bit isolated from the rest of this work, at least on the first glimpse. In Theorem 5.2.24, we have seen that the combination of Proposition 2.3.4 with the vector-valued version of Shen's L^p-extrapolation theorem, Theorem 3.1.2, provides a new method to prove maximal L^q-regularity of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$. This chapter shows, that this method is not even applicable for the Stokes operator on bounded Lipschitz domains, but also for higher-order elliptic systems in divergence form on rough domains.

More precisely, the main object under consideration is an elliptic operator A in divergence form of order $2m$, formally given by

$$(Au)_i = (-1)^m \sum_{j=1}^{N} \sum_{|\alpha|,|\beta|=m} \partial^\alpha [\mu^{ij}_{\alpha\beta} \partial^\beta u_j] \qquad (1 \leq i \leq N)$$

on a bounded domain $\Omega \subset \mathbb{R}^d, d \geq 2$. The coefficients are supposed to be essentially bounded and complex-valued; ellipticity is enforced by a Gårding type inequality. Each component of u is supposed to satisfy mixed Dirichlet/Neumann boundary conditions on possibly different portions of the boundary. That is to say, on given closed subsets $D_i \subset \partial\Omega$ all derivatives of order less than $m-1$ of the ith component of $u \in \mathcal{D}(A)$ are assumed to vanish and on its complement relative to $\partial\Omega$ the ith component is assumed to satisfy homogeneous Neumann boundary conditions arising naturally by the definition of the operator. If all D_i coincide, a detailed introduction to higher-order elliptic operators and a discussion on these Neumann boundary conditions is included in Brewster, D. Mitrea, I. Mitrea, and M. Mitrea [12, Sec. 7]. The given boundary conditions have an impact on the admissible geometric constellation of $\partial\Omega$, namely every point in $\overline{\partial\Omega \setminus [\cap_{i=1}^N D_i]}$ is assumed to possess a bi-Lipschitzian coordinate chart. Thus, we record that the intersection of the sets D_i is free from further assumptions. We emphasize that the results include the pure Dirichlet and Neumann cases, so that in the first case, it suffices to assume the sole openness of Ω and in the second case that Ω is a bounded Lipschitz domain. Note that in comparison to the chapters before, the definition of a bounded Lipschitz domain is generalized in this chapter, see Assumption 7.1.1.

The strategy to establish the weak reverse Hölder estimates is simpler, as in the case of the Stokes resolvent problem. Note that for the Stokes resolvent problem, a detailed investigation of the L^2-Dirichlet problem was needed, whose resolution required a study of the L^2-Dirichlet problem for the Stokes system (with resolvent parameter $\lambda = 0$). In the elliptic situation, the verification of the weak reverse Hölder estimates is easier, as the following example for harmonic functions shows. Assume that

$$\lambda u - \Delta u = 0 \qquad \text{in } B(0, 3r).$$

By Sobolev's embedding theorem, we find

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{B(0,r)} |u|^{\frac{2d}{d-2}}\,\mathrm{d}x\Big)^{\frac{d-2}{2d}} \leq C\bigg\{\Big(\frac{1}{r^d}\int_{B(0,r)} |u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ + r\Big(\frac{1}{r^d}\int_{B(0,r)} |\nabla u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\}.\end{aligned}$$

Next, use Caccioppoli's inequality to deduce

$$r\Big(\frac{1}{r^d}\int_{B(0,r)} |\nabla u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \leq C\Big(\frac{1}{r^d}\int_{B(0,2r)} |u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

This proves the validity of weak reverse Hölder estimates for solutions of the homogeneous scalar Helmholtz equation. Transferring this approach to a square function term of the form $[\sum_{n=1}^{n_0} |\lambda_n|^2 \, |u_n|^2]^{1/2}$, where λ_n and u_n solve the scalar Helmholtz equation above, we record that it suffices to prove the following two steps.

(1) Prove a Sobolev inequality of the type

$$\begin{aligned}\Bigg(\frac{1}{r^d}\int_{B(0,r)}\Big[\sum_{n=1}^{n_0}|\lambda_n|^2\,|u_n|^2\Big]^{\frac{d}{d-2}}\,\mathrm{d}x\Bigg)^{\frac{d-2}{2d}} &\\ \leq C\Bigg\{\Bigg(\frac{1}{r^d}\int_{B(0,r)}\sum_{n=1}^{n_0}|\lambda_n|\,|u_n|^2\,\mathrm{d}x\Bigg)^{\frac{1}{2}} &\\ + r\Bigg(\frac{1}{r^d}\int_{B(0,r)}\sum_{n=1}^{n_0}|\lambda_n|^2\,|\nabla u_n|^2\,\mathrm{d}x\Bigg)^{\frac{1}{2}}\Bigg\}&,\end{aligned}$$

with C being independent of n_0, λ_n, u_n, and r.

(2) Establish Caccioppoli's inequality for solutions to the homogeneous higher-order elliptic resolvent equation.

Clearly, at the boundary, more technicalities will arise, so that we allow for the Sobolev inequality, that the domain of integration can blow-up by a constant factor. Moreover, the verification of Caccioppoli's inequality is quite technical and long. For this, we generalize the proof of Barton [8, Sec. 3] to mixed boundary conditions. Moreover, we will see that for elliptic systems of order $2m$, Caccioppoli's inequality gives an estimate for all derivatives of order less or equal than m by the L^2-norm of the solution. Hence, we can iterate the Sobolev inequality m times, what leads to a larger Lebesgue exponent on the left-hand side of the weak reverse Hölder estimates.

The main result of this chapter is Theorem 7.2.4.

7.1 The geometric setup

The geometric assumptions, that are required for the proof are the following.

Assumption 7.1.1. *The domain $\Omega \subset \mathbb{R}^d$ is bounded and there exists a possibly empty, closed set $D \subset \partial\Omega$, such that for every point $x \in \overline{\partial\Omega \setminus D}$ there exists a bi-Lipschitz coordinate chart. More precisely, there exists a number $M \geq 1$ such that for every $x \in \overline{\partial\Omega \setminus D}$ there exists an open neighborhood $U_x \subset \mathbb{R}^d$ and a bi-Lipschitz homeomorphism $\Phi_x : U_x \to (-1,1)^d$ with Lipschitz constants of Φ_x and Φ_x^{-1} being bounded by M and fulfilling the mapping properties*

$$\begin{aligned} \Phi_x(x) &= 0 \\ \Phi_x(\Omega \cap U_x) &= (-1,1)^{d-1} \times (0,1) \\ \Phi_x(\partial\Omega \cap U_x) &= (-1,1)^{d-1} \times \{0\}. \end{aligned}$$

For the rest of this chapter, we will use the notation, that $Q(x,r)$ denotes the cubes in $\mathbb{R}^d$ with center x, with diameter $2r$, and faces parallel to the coordinate axes.

Remark 7.1.2. (1) If $D = \emptyset$ in Assumption 7.1.1, then Ω is called a Lipschitz domain. However, note that this definition of a Lipschitz domain deviates from the one given in Definition 1.3.1 and is more general. See Grisvard [45, p. 8f] for an example of a domain that satisfies the premises of Assumption 7.1.1 with $D = \emptyset$, but not of Definition 1.3.1.

(2) For $y \in \partial\Omega \cap U_x$ with $|x - y| \leq 1/(2M)$ and $0 < r \leq 1/4$ it is easy to see that $Q(\Phi_x(y), r) \subset (-1,1)^d$ holds. Denote the bi-Lipschitzian counterpart of this cube by $U_{y,r} := \Phi_x^{-1}((Q(\Phi_x(y), r)))$ and denote its portion in Ω by $U_{y,r}^+ := U_{y,r} \cap \Omega$. Note that the bi-Lipschitz property of Φ_x implies that for $0 < s < t \leq 1$

$$(\sqrt{d}M)^{-1}(t-s)r \leq \operatorname{dist}(\partial U_{y,sr}, \partial U_{y,tr}) \leq \frac{M}{\sqrt{d}}(t-s)r$$

holds.

(3) With y and r as in (2), the bi-Lipschitzianity of Φ_x implies

$$B(y, r/(M\sqrt{d})) \subset U_{y,r} \subset B(y, Mr).$$

For further reference, we record the following proposition dealing with local extensions at the Lipschitz boundary of Ω.

Proposition 7.1.3. *Let Ω be a domain subject to Assumption 7.1.1. Let $x \in \overline{\partial\Omega \setminus D}$, and y and r be as in Remark 7.1.2 (2). Then there exists a bounded extension operator $\mathcal{E}_{y,r} : \mathrm{L}^1(U_{y,r}^+) \to \mathrm{L}^1(U_{y,r})$, i.e., $\mathcal{E}_{y,r} u|_{U_{y,r}^+} = u$, which restricts for all $p \in [1, \infty)$ to a bounded operator from $\mathrm{L}^p(U_{y,r}^+)$ into $\mathrm{L}^p(U_{y,r})$ and from $\mathrm{W}^{1,p}(U_{y,r}^+)$ into $\mathrm{W}^{1,p}(U_{y,r})$. Moreover, on $U_{y,r} \setminus U_{y,r}^+$ the function $\mathcal{E}_{y,r} u$ is given by $u \circ \psi$, where $\psi := \Phi_x^{-1} \circ \mathfrak{R} \circ \Phi_x$ and $\mathfrak{R}$ is the reflection at the upper half-space boundary. Furthermore, ψ is Lipschitz continuous on $U_{y,r} \setminus U_{y,r}^+$ with Lipschitz constant bounded by M^2.*

Proof. The proposition easily follows by transforming u by Φ_x^{-1} to the cube $Q(\Phi_x(y), r)$, performing an even reflection at the upper half-space boundary, and transforming the resulting function back to $U_{y,r}$ using Φ_x. □

If Ω and D are subject to Assumption 7.1.1, notice that for each $x \in \overline{\partial\Omega \setminus D}$ the sets $U_x \cap \Omega$ are (ε, δ)-domains in the sense of Jones [56]. This follows as the (ε, δ)-property is preserved under bi-Lipschitz homeomorphisms and since $(-1, 1)^{d-1} \times (0, 1)$ is an (ε, δ)-domain. For a proof of this fact, we refer to Egert [24, Lem. 2.2.20]. Using this covering property of the boundary strip near $\overline{\partial\Omega \setminus D}$ by (ε, δ)-domains, one can use a partition of unity in order to obtain Sobolev extension operators for the Sobolev spaces $\mathrm{W}^{m,2}_D(\Omega)$ adapted to mixed boundary conditions introduced in Definition 1.1.11. Such a construction yielding semi-universal extension operators can be found in Brewster, D. Mitrea, I. Mitrea, and M. Mitrea [12, Thm. 3.9]. More precisely, these authors prove that there exists a linear operator $\mathcal{E}$ mapping locally integrable functions on Ω into Lebesgue measurable functions on $\mathbb{R}^d$, which satisfies $\mathcal{E}u|_\Omega = u$, which is bounded from $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2(\mathbb{R}^d)$, and which is bounded from $\mathrm{W}^{k,2}_D(\Omega)$ into $\mathrm{W}^{k,2}_D(\mathbb{R}^d)$ for all $1 \leq k \leq m$. Moreover, in the present situation the operator norms depend only on d, M, and m. Using this, we can prove the following proposition.

Proposition 7.1.4. *Let Ω and D be subject to Assumption 7.1.1. Then, for each $m \in \mathbb{N}$ there exists a constant $C > 0$ depending only on d, M, and m such that*

$$\|u\|_{\mathrm{W}^{m,2}(\Omega)} \leq C \Big[\|u\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla^m u\|^2_{\mathrm{L}^2(\Omega)} \Big]^{1/2} \qquad (u \in \mathrm{W}^{m,2}_D(\Omega)).$$

Proof. We perform an induction on m. Note that there is nothing to do in the case $m = 1$ so that we can directly perform the induction step. Thus, assume the validity of the statement above for a fixed number $m \in \mathbb{N}$. Then there exists a constant $K \geq 1$ such that

$$\|u\|_{\mathrm{W}^{m+1,2}(\Omega)} \leq K\Big[\|u\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla^m u\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla^{m+1} u\|_{\mathrm{L}^2(\Omega)}^2\Big]^{1/2}.$$

Next, use the Gagliardo–Nirenberg inequality, see Nirenberg [78, p. 125], to deduce

$$\|\nabla^m u\|_{\mathrm{L}^2(\Omega)}^2 \leq \|\nabla^m \mathcal{E} u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 \leq C\|\mathcal{E} u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\frac{2}{m+1}} \|\nabla^{m+1} \mathcal{E} u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\frac{2m}{m+1}}.$$

Using the boundedness properties of $\mathcal{E}$ discussed in the paragraph prior to the proposition shows that

$$\|\nabla^m u\|_{\mathrm{L}^2(\Omega)}^2 \leq C\|u\|_{\mathrm{L}^2(\Omega)}^{\frac{2}{m+1}} \|u\|_{\mathrm{W}^{m+1,2}(\Omega)}^{\frac{2m}{m+1}}.$$

Employing Young's inequality yields

$$\leq \frac{K^{2m} C^{m+1}}{m+1} \|u\|_{\mathrm{L}^2(\Omega)}^2 + \frac{m}{K^2(m+1)} \|u\|_{\mathrm{W}^{m+1,2}(\Omega)}^2.$$

Finally, the induction hypothesis delivers

$$\begin{aligned} &\leq \frac{K^{2m} C^{m+1} + m}{m+1} \|u\|_{\mathrm{L}^2(\Omega)}^2 + \frac{m}{m+1} \|\nabla^m u\|_{\mathrm{L}^2(\Omega)}^2 \\ &\quad + \frac{m}{m+1} \|\nabla^{m+1} u\|_{\mathrm{L}^2(\Omega)}^2. \end{aligned}$$

Absorbing the second summand on the right-hand side to the left-hand side concludes the induction step. □

7.2 The operator

We fix some notation quantifying the elliptic system under consideration, that will be used throughout the chapter.

Let $N \in \mathbb{N}$ denote the number of equations of the elliptic system, which itself is supposed to be of order $2m$ with $m \in \mathbb{N}$. Fix $\Omega \subset \mathbb{R}^d$ and closed sets $D_1, \dots, D_N \subset \partial\Omega$ and define

$$D := \bigcap_{i=1}^{N} D_i.$$

Suppose that Ω and D fulfill Assumption 7.1.1. Define

$$\mathbb{D} := (D_1, \dots, D_N).$$

Recall the Sobolev spaces $\mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)$ adapted to mixed boundary conditions defined in Definition 1.1.11. We will refer to the sets $D_1, \dots, D_N$ as the Dirichlet part.

For the coefficients $\mu^{ij}_{\alpha\beta}$ of the elliptic system, we make the following assumption.

Assumption 7.2.1. *The coefficients $\mu^{ij}_{\alpha\beta} : \Omega \to \mathbb{C}$, $1 \leq i, j \leq N$, $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| = |\beta| = m$ are Lebesgue measurable, essentially bounded functions, with bound $\Lambda > 0$, such that the associated sesquilinear form*

$$\mathfrak{a} : \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N) \times \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N),$$
$$(u, v) \mapsto \sum_{i,j=1}^{N} \sum_{|\alpha|,|\beta|=m} \int_{\Omega} \mu^{ij}_{\alpha\beta} \partial^{\beta} u_j \overline{\partial^{\alpha} v_i} \, \mathrm{d}x$$

is elliptic in the sense that for some $\kappa > 0$ and all $u \in \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)$ Gårding's inequality

$$\mathrm{Re}(\mathfrak{a}(u, u)) \geq \kappa \sum_{i=1}^{N} \sum_{|\alpha|=m} \int_{\Omega} |\partial^{\alpha} u_i|^2 \, \mathrm{d}x$$

is valid.

Remark 7.2.2. Under Assumption 7.2.1 the sesquilinear form $\mathfrak{a}$ is sectorial of an angle $\omega \in [0, \frac{\pi}{2})$, i.e., the *numerical range*

$$\{\mathfrak{a}(u, u) : u \in \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)\}$$

is contained in a sector $\overline{\mathrm{S}_{\omega}}$.

Since $\mathfrak{a}$ is densely defined on $\mathrm{L}^2(\Omega; \mathbb{C}^N)$, sectorial, and closed (due to ellipticity of $\mathfrak{a}$ and Proposition 7.1.4) it is known from classical form theory, see Kato [58, Thm. VI.2.1], that there exists a unique sectorial operator A on $\mathrm{L}^2(\Omega; \mathbb{C}^N)$ of angle $\omega \in [0, \frac{\pi}{2})$ such that $\mathcal{D}(A) \subset \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)$ and

$$\mathfrak{a}(u, v) = \int_\Omega \langle Au, v \rangle \, \mathrm{d}x \qquad (u \in \mathcal{D}(A), v \in \mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)). \tag{7.1}$$

If B is a linear operator on $\mathrm{L}^2(\Xi; \mathbb{C}^N)$ on a bounded domain $\Xi \subset \mathbb{R}^d$ and $p > 2$, define the L^p-realization of B_p as the part of B in $\mathrm{L}^p(\Xi; \mathbb{C}^N)$, i.e.,

$$\begin{aligned} \mathcal{D}(B_p) &:= \{u \in \mathcal{D}(B) \cap \mathrm{L}^p(\Xi; \mathbb{C}^N) : Bu \in \mathrm{L}^p(\Xi; \mathbb{C}^N)\}, \\ B_p u &:= Bu \qquad (u \in \mathcal{D}(B_p)). \end{aligned}$$

If $p < 2$ define B_p as the closure of B in $\mathrm{L}^p(\Xi; \mathbb{C}^N)$, if it exists. We record the following proposition, which connects for densely defined operators B the operators B_p and $(B^*)_{p'}$, where p' is the Hölder conjugate exponent of p and B^* the Hilbert space adjoint of B.

This proposition is similar to Proposition 5.2.16 in the context of the Stokes operator.

Lemma 7.2.3. *Let $\Xi \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, $p \in (2, \infty)$ and let B be a densely defined operator on $\mathrm{L}^2(\Xi; \mathbb{C}^N)$. Then $\mathcal{D}(B)$ is dense in $\mathrm{L}^{p'}(\Xi; \mathbb{C}^N)$ and B is closable in $\mathrm{L}^{p'}(\Xi; \mathbb{C}^N)$ if and only if the part of B^* in $\mathrm{L}^p(\Xi; \mathbb{C}^N)$ is densely defined. In this case the identity $(B_{p'})^* = (B^*)_p$ holds true.*

Proof. First, assume there exists $f \in \mathrm{L}^p(\Xi; \mathbb{C}^N)$ such that

$$\int_\Xi \langle u, f \rangle \, \mathrm{d}x = 0 \qquad (u \in \mathcal{D}(B)).$$

By the boundedness of Ξ, we find $f \in \mathrm{L}^2(\Xi; \mathbb{C}^N)$, and by the density of $\mathcal{D}(B)$ in $\mathrm{L}^2(\Xi; \mathbb{C}^N)$, it follows that f must be zero. Consequently, $\mathcal{D}(B)$ is dense in $\mathrm{L}^{p'}(\Xi; \mathbb{C}^N)$.

Let B be closable in $\mathrm{L}^{p'}(\Xi; \mathbb{C}^N)$. Because $\mathcal{D}(B) \subset \mathcal{D}(B_{p'})$ the closure of B is densely defined and by definition of its domain, for each $u \in \mathcal{D}(B_{p'})$

there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(B)$ with $u_n \to u$ in $\mathrm{L}^{p'}(\Xi;\mathbb{C}^N)$ and $Bu_n \to B_{p'}u$ in $\mathrm{L}^{p'}(\Xi;\mathbb{C}^N)$. Thus, for $v \in \mathcal{D}((B^*)_p)$, we find

$$\int_\Xi \langle B_{p'}u, v\rangle \,\mathrm{d}x = \lim_{n\to\infty} \int_\Xi \langle u_n, B^*v\rangle \,\mathrm{d}x = \int_\Xi \langle u, (B^*)_p v\rangle \,\mathrm{d}x.$$

We derive the inclusion $\mathcal{D}((B^*)_p) \subset \mathcal{D}((B_{p'})^*)$ and equality of the operators on $\mathcal{D}((B^*)_p)$.

If $w \in \mathcal{D}((B_{p'})^*)$, we find for $u \in \mathcal{D}(B) \subset \mathcal{D}(B_{p'})$

$$\int_\Xi \langle Bu, w\rangle \,\mathrm{d}x = \int_\Xi \langle u, (B_{p'})^* w\rangle \,\mathrm{d}x.$$

Consequently, $w \in \mathcal{D}(B^*) \cap \mathrm{L}^p(\Xi;\mathbb{C}^N)$ and $B^*w = (B_{p'})^*w \in \mathrm{L}^p(\Xi;\mathbb{C}^N)$ so that $w \in \mathcal{D}((B^*)_p)$. This proves $\mathcal{D}((B^*)_p) = \mathcal{D}((B_{p'})^*)$, so that the part of B^* in $\mathrm{L}^p(\Xi;\mathbb{C}^N)$ is densely defined by Schechter [85, Thm. 7.20 & Lem. 7.21].

Now assume that $(B^*)_p$ is densely defined and let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(B)$ be a sequence with $u_n \to 0$ in $\mathrm{L}^{p'}(\Xi;\mathbb{C}^N)$ such that $(Bu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^{p'}(\Xi;\mathbb{C}^N)$ with limit f. Then

$$\int_\Xi \langle f, v\rangle \,\mathrm{d}x = \lim_{n\to\infty} \int_\Xi \langle u_n, (B^*)_p v\rangle \,\mathrm{d}x = 0 \qquad (v \in \mathcal{D}((B^*)_p)).$$

Hence, f is zero by density of $\mathcal{D}((B^*)_p)$ in $\mathrm{L}^p(\Xi;\mathbb{C}^N)$ so that B is closable in $\mathrm{L}^{p'}(\Xi;\mathbb{C}^N)$. □

With this lemma in mind, we can formulate the main result of this chapter. For this, let $\omega \in [0, \frac{\pi}{2})$ denote the angle of sectoriality of A on $\mathrm{L}^2(\Omega;\mathbb{C}^N)$.

Theorem 7.2.4. *Let $N, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be a bounded domain, and $D_1, \dots, D_N \subset \partial\Omega$ be closed sets. Define*

$$D := \bigcap_{i=1}^N D_i$$

and assume that Ω and D fulfill Assumption 7.1.1. Let the coefficients $\mu^{ij}_{\alpha\beta}$, $1 \le i,j \le N$, $|\alpha| = |\beta| = m$ be subject to Assumption 7.2.1, and let A be the elliptic operator of order $2m$ as defined in (7.1).

Then for each $\theta \in [\omega, \pi]$ there exists $\varepsilon \geq 0$ with $\varepsilon > 0$ if $\theta \in (\omega, \pi]$, depending only on d, M, m, N, κ, θ, ω, and Λ, such that the following statement is valid.

If $2m < d$, then for all p satisfying

$$\frac{2d}{d+2m} - \varepsilon < p < \frac{2d}{d-2m} + \varepsilon$$

or, if $2m \geq d$, then for all $p \in (1, \infty)$, the part of A in $\mathrm{L}^p(\Omega; \mathbb{C}^N)$ is densely defined (if $p > 2$) and A is closable in $\mathrm{L}^p(\Omega; \mathbb{C}^N)$ (if $p < 2$). Moreover, A_p is sectorial of angle θ and for every $\theta' \in (\theta, \pi]$ the family of operators $\{\lambda(\lambda + A_p)^{-1}\}_{\lambda \in \mathrm{S}_{\pi-\theta'}}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\mathrm{L}^p(\Omega; \mathbb{C}^N))$.

Furthermore, for every $1 < q < \infty$ and p in the range above, A_p has maximal L^q-regularity.

Remark 7.2.5. To prove this theorem, we can reduce matters to the case $p > 2$. Indeed, note that Assumption 7.2.1 on the coefficients is stable under the operation $\mu^{ij}_{\alpha\beta} \mapsto \overline{\mu^{ji}_{\beta\alpha}}$ so that if the theorem is proven under this assumption for the L^p-realization of A and $p > 2$, it then is also proven for the L^p-realization of A^*. For the situation of $p < 2$ one can argue by duality using Remark 2.3.2 (3).

As the case $p > 2$ remains, it is desirable to appeal to the vector-valued version of Shen's L^p-extrapolation theorem, which requires vector-valued weak reverse Hölder estimates.

7.3 Vector-valued weak reverse Hölder estimates

We begin by proving Caccioppoli's inequality for higher-order elliptic systems subject to mixed boundary conditions. The proof is essentially the one of Barton [8, Sec. 3], with the modification that we not just consider balls, but also the sets $U^+_{y,r}$ defined in Remark 7.1.2 and solutions which locally satisfy

$$\lambda u_i + (-1)^m \sum_{j=1}^{N} \sum_{|\alpha|,|\beta|=m} \partial^\alpha [\mu^{ij}_{\alpha\beta} \partial^\beta u_j] = 0 \qquad (1 \leq i \leq N).$$

Barton considered only the case $\lambda = 0$.

As we are now concerned with the proof of Theorem 7.2.4, we will assume that Ω and D are subject to Assumption 7.1.1. We will also use λ as a resolvent parameter, so $\lambda \in \mathrm{S}_{\pi-\theta}$, where $\theta \in (\omega, \pi]$ and ω is such that A is sectorial of angle ω on $\mathrm{L}^2(\Omega; \mathbb{C}^N)$. Recall that M is the bound for the bi-Lipschitz constants of the homeomorphisms Φ_x, see Assumption 7.1.1.

Lemma 7.3.1 (Caccioppoli's inequality part 1)**.** *Let $x_0 \in \overline{\Omega}$ and $r > 0$. Distinguish the following cases:*

(1) $x_0 \in \Omega$ and $r < \operatorname{dist}(x_0, \partial\Omega)$;

(2) $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$ and $r \leq 1/4$;

(3) $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) > 1/(2M)$ and $r \leq 1/(2M)$.

Let $0 < s < t \leq 1$ and $f \in \mathrm{L}^2(\Omega; \mathbb{C}^N)$ be such that $f = 0$ on $B(x_0, r) \cap \Omega$ in cases (1) *and* (3) *or $f = 0$ on $U^+_{x_0,r}$ in case* (2)*. Define $u := (\lambda + A)^{-1} f$. Then there exists a constant $C > 0$ depending only on d, N, m, κ, M, θ, ω, and Λ, such that*

$$|\lambda| \int_{\mathcal{B}_{sr} \cap \Omega} |u|^2 \, \mathrm{d}x + \int_{\mathcal{B}_{sr} \cap \Omega} |\nabla^m u|^2 \, \mathrm{d}x \leq C \sum_{k=0}^{m-1} [(t-s)r]^{-2(m-k)} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} |\nabla^k u|^2 \, \mathrm{d}x,$$

where in cases (1) *and* (3)*, $\mathcal{B}_{\alpha r} := B(x_0, \alpha r)$, and in case* (2)*, $\mathcal{B}_{\alpha r} := U_{x_0, \alpha r}$ for $\alpha \in (0, 1]$.*

Proof. Take a cutoff function $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ which in cases (1) and (3) is identically one on $B(x_0, sr)$ and zero on $B(x_0, tr)^c$ and which satisfies $\|\nabla^k \varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathbb{R}^{d^k})} \leq C_d [(t-s)r]^{-k}$ for all $0 \leq k \leq m$. In case (2), take φ to be one on $U_{x_0, sr}$ and zero on $U^c_{x_0, tr}$ with estimates $\|\nabla^k \varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathbb{R}^{d^k})} \leq C_{d,M} [(t-s)r]^{-k}$ for all $0 \leq k \leq m$. In case (2), such a function φ exists by the estimates proven in Remark 7.1.2 (2) combined with McLean [70, Thm. 3.6]. The constant C_d depends only on d and $C_{d,M}$ on d and M.

Define $\psi := u \varphi^{2m}$, which again is a function in $\mathrm{W}^{m,2}_{\mathbb{D}}(\Omega; \mathbb{C}^N)$, since φ is smooth. Testing with ψ yields

$$0 = \lambda \int_{\mathcal{B}_{tr} \cap \Omega} |u \varphi^m|^2 \, \mathrm{d}x + \mathfrak{a}(u, \psi).$$

Moreover, Leibniz' rule yields numbers $c_{\alpha\beta} \in \mathbb{N}$ with $c_{\alpha 0} = c_{\alpha\alpha} = 1$ such that

$$\mathfrak{a}(u, \psi) = \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \sum_{\gamma<\alpha} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^{\beta} u_j c_{\alpha\gamma} \partial^{\alpha-\gamma} \varphi^m \partial^{\gamma} [\varphi^m \overline{u_i}] \, \mathrm{d}x$$
$$+ \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \int_{\mathcal{B}_{tr} \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^{\beta} u_j \varphi^m \partial^{\alpha} [\varphi^m \overline{u_i}] \, \mathrm{d}x.$$

Note that the integration in the first integral on the right-hand side is performed only on $[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega$ since φ^m is constant on both $\mathcal{B}_{sr}$ and $\mathcal{B}_{tr}^c$. Next, we show that

$$\sum_{\gamma<\alpha} c_{\alpha\gamma} \partial^{\alpha-\gamma} \varphi^m \partial^{\gamma} [\varphi^m \overline{u_i}] = \sum_{\delta<\alpha} \varphi^m \zeta_{\alpha\delta} \partial^{\delta} \overline{u_i} \tag{7.2}$$

with $\|\zeta_{\alpha\delta}\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C_{d,M,m} [(t-s)r]^{|\delta|-|\alpha|}$, where $C_{d,M,m}$ solely depends on d, M, and m. Indeed, by means of Leibniz' formula

$$\sum_{\gamma<\alpha} c_{\alpha\gamma} \partial^{\alpha-\gamma} \varphi^m \partial^{\gamma} [\varphi^m \overline{u_i}] = \sum_{\gamma<\alpha} c_{\alpha\gamma} \partial^{\alpha-\gamma} \varphi^m \sum_{\delta\leq\gamma} c_{\gamma\delta} \partial^{\gamma-\delta} \varphi^m \partial^{\delta} \overline{u_i}.$$

Interchanging the sums yields

$$= \sum_{\delta<\alpha} \sum_{\delta\leq\gamma<\alpha} c_{\alpha\gamma} c_{\gamma\delta} \partial^{\alpha-\gamma} \varphi^m \partial^{\gamma-\delta} \varphi^m \partial^{\delta} \overline{u_i}.$$

Next, for fixed $\delta < \alpha$ and any $\delta \leq \gamma < \alpha$, consider

$$\partial^{\alpha-\gamma} \varphi^m \partial^{\gamma-\delta} \varphi^m.$$

Notice by the product rule, that $\partial^{\alpha-\gamma}\varphi^m$ is a sum of terms of the form

$$\prod_{e=1}^{E} [\partial^{\iota_e} \varphi] \varphi^l,$$

where $1 \leq E \leq |\alpha - \gamma|$, $\iota_e \in \mathbb{N}_0^d$ with $\sum_{e=1}^{E} |\iota_e| = |\alpha - \gamma|$, and $l := m - E$. With the obvious modifications, the same is clearly valid for $\partial^{\gamma-\delta}\varphi^m$. Consequently, a generic term of the product $\partial^{\alpha-\gamma}\varphi^m \partial^{\gamma-\delta}\varphi^m$ is of the form

$$\prod_{e=1}^{E_1} [\partial^{\iota_e} \varphi] \varphi^{l_1} \prod_{e=1}^{E_2} [\partial^{\kappa_e} \varphi] \varphi^{l_2}$$

with $1 \leq E_1 \leq |\alpha - \gamma|$, $0 \leq E_2 \leq |\gamma - \delta|$, $\iota_e, \kappa_e \in \mathbb{N}_0^d$, $\sum_{e=1}^{E_1} |\iota_e| = |\alpha - \gamma|$, $\sum_{e=1}^{E_2} |\kappa_e| = |\gamma - \delta|$, $l_1 = m - E_1$, and $l_2 = m - E_2$. Since

$$l_1 + l_2 = 2m - E_1 - E_2 \geq 2m - |\alpha - \gamma| - |\gamma - \delta| = 2m - (|\alpha| - |\delta|) \geq m$$

and

$$\sum_{e=1}^{E_1} |\iota_e| + \sum_{e=1}^{E_2} |\kappa_e| = |\alpha| - |\delta|$$

the claim is proved. Using (7.2), we derive

$$\begin{aligned} \mathfrak{a}(u, \psi) = &\sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \sum_{\delta<\alpha} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^\beta u_j \varphi^m \zeta_{\alpha\delta} \partial^\delta \overline{u_i} \, \mathrm{d}x \\ &+ \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \int_{\mathcal{B}_{tr} \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^\beta u_j \varphi^m \partial^\alpha [\varphi^m \overline{u_i}] \, \mathrm{d}x. \end{aligned}$$

Rewriting $\partial^\beta u_j \varphi^m$ by using Leibniz' rule reveals

$$\begin{aligned} = &- \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \sum_{\delta<\alpha} \sum_{\gamma<\beta} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} \mu_{\alpha\beta}^{ij} c_{\beta\gamma} \partial^\gamma u_j \partial^{\beta-\gamma} \varphi^m \zeta_{\alpha\delta} \partial^\delta \overline{u_i} \, \mathrm{d}x \\ &- \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \sum_{\gamma<\beta} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} \mu_{\alpha\beta}^{ij} c_{\beta\gamma} \partial^\gamma u_j \partial^{\beta-\gamma} \varphi^m \partial^\alpha [\varphi^m \overline{u_i}] \, \mathrm{d}x \\ &+ \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \sum_{\delta<\alpha} \int_{[\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^\beta [\varphi^m u_j] \zeta_{\alpha\delta} \partial^\delta \overline{u_i} \, \mathrm{d}x \\ &+ \sum_{|\alpha|,|\beta|=m} \sum_{i,j=1}^{N} \int_{\mathcal{B}_{tr} \cap \Omega} \mu_{\alpha\beta}^{ij} \partial^\beta [\varphi^m u_j] \partial^\alpha [\varphi^m \overline{u_i}] \, \mathrm{d}x. \end{aligned}$$

Note that the last term on the right-hand side can be identified with $\mathfrak{a}(\varphi^m u, \varphi^m u)$. Summarizing, we find a constant $C > 0$ depending only on d, N, m, M, and Λ such that

$$\begin{aligned} &\left| \lambda \int_{\mathcal{B}_{tr} \cap \Omega} |\varphi^m u|^2 \, \mathrm{d}x + \mathfrak{a}(\varphi^m u, \varphi^m u) \right| \\ &\leq C \Bigg\{ \sum_{|\alpha|,|\beta|=m} \sum_{\delta<\alpha} \sum_{\gamma<\beta} \frac{\|\partial^\gamma u\|_{\mathrm{L}^2([\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega; \mathbb{C}^N)}}{[(t-s)r]^{m-|\gamma|}} \frac{\|\partial^\delta u\|_{\mathrm{L}^2([\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega; \mathbb{C}^N)}}{[(t-s)r]^{m-|\delta|}} \\ &\qquad + \sum_{|\beta|=m} \sum_{\gamma<\beta} \frac{\|\partial^\gamma u\|_{\mathrm{L}^2([\mathcal{B}_{tr} \setminus \mathcal{B}_{sr}] \cap \Omega; \mathbb{C}^N)}}{[(t-s)r]^{m-|\gamma|}} \|\nabla^m [\varphi^m u]\|_{\mathrm{L}^2(\mathcal{B}_{tr} \cap \Omega; \mathbb{C}^{Nd^m})} \Bigg\}. \end{aligned}$$

Using the sectoriality of $\mathfrak{a}$, see Remark 7.2.2, as well as $\lambda \in \mathrm{S}_{\pi-\theta}$ and $\pi - \theta + \omega < \pi$, we can use Lemma 5.2.4 to conclude that there exists a constant $C_{\theta,\omega}$ depending only on θ and ω such that

$$\begin{aligned}\Big|\lambda \int_{\mathcal{B}_{tr}\cap\Omega} |\varphi^m u|^2 \,\mathrm{d}x + \mathfrak{a}(\varphi^m u, \varphi^m u)\Big| \\ \geq C_{\theta,\omega}\Big\{ |\lambda| \int_{\mathcal{B}_{tr}\cap\Omega} |\varphi^m u|^2 \,\mathrm{d}x + |\mathfrak{a}(\varphi^m u, \varphi^m u)|\Big\}\end{aligned}$$

holds. By Gårding's inequality, we conclude

$$\geq C_{\theta,\omega}\Big\{ |\lambda| \int_{\mathcal{B}_{tr}\cap\Omega} |\varphi^m u|^2 \,\mathrm{d}x + \kappa \int_{\mathcal{B}_{tr}\cap\Omega} |\nabla^m[\varphi^m u]|^2 \,\mathrm{d}x\Big\}.$$

Next, use Young's inequality to estimate

$$\begin{aligned} C \sum_{|\beta|=m}\sum_{\gamma<\beta} \frac{\|\partial^\gamma u\|_{\mathrm{L}^2([\mathcal{B}_{tr}\setminus\mathcal{B}_{sr}]\cap\Omega)}}{[(t-s)r]^{m-|\gamma|}} \|\nabla^m[\varphi^m u]\|_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^m})} \\ \leq \frac{C}{2\varepsilon} \sum_{|\beta|=m}\sum_{\gamma<\beta} \frac{\|\partial^\gamma u\|^2_{\mathrm{L}^2([\mathcal{B}_{tr}\setminus\mathcal{B}_{sr}]\cap\Omega;\mathbb{C}^N)}}{[(t-s)r]^{2(m-|\gamma|)}} \\ + \frac{C\varepsilon}{2} \sum_{|\beta|=m}\sum_{\gamma<\beta} \|\nabla^m[\varphi^m u]\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^m})}.\end{aligned}$$

Choose ε, such that

$$\frac{C\varepsilon}{2} \sum_{|\beta|=m}\sum_{\gamma<\beta} 1 = \frac{C_{\theta,\omega}\kappa}{2}.$$

Then, absorb $C_{\theta,\omega}\kappa\|\nabla^m[\varphi^m u]\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^m})}/2$ from the right-hand side onto the left-hand side of the whole inequality. Using that and $\varphi = 1$ on $\mathcal{B}_{sr}$ concludes the proof. $\square$

The preceding lemma shows that one can locally control $|\lambda|^{1/2} u$ and $\nabla^m u$ in L^2 by the L^2-norms of all derivatives of order strictly less than m. However, it is desirable to control them solely by u in the L^2-norm. To prove that, we adapt the proof of Barton [8, Thm. 18] to mixed boundary conditions. For this purpose, we prove the following lemma, which is a generalization of Giaquinta and Martinazzi [37, Lem. 8.18] and is implicitly contained in the proof of Barton.

Lemma 7.3.2. *Let $0 \leq s_0 < t_0 < \infty$ and $k \in \mathbb{N}$. Assume that $\phi : [s_0, t_0] \to \mathbb{R}$ is a non-negative bounded function. Suppose that there exist constants $A_1, \dots, A_k > 0$, $\alpha_1, \dots, \alpha_k > 0$, and $0 \leq \varepsilon < 1$ such that for all $s_0 \leq s < t \leq t_0$ we have*

$$\phi(s) \leq \sum_{l=1}^{k} A_l (t-s)^{-\alpha_l} + \varepsilon \phi(t).$$

Then there exists a constant $C > 0$ depending only on $\max_{i=1}^{k}\{\alpha_i\}$ and ε, such that for all $s_0 \leq s < t \leq t_0$ we have

$$\phi(s) \leq C \sum_{l=1}^{k} A_l (t-s)^{-\alpha_l}.$$

Proof. Let $0 < \tau < 1$ to be determined and define

$$\rho_0 := s, \quad \rho_{n+1} := \rho_n + (1-\tau)\tau^n (t-s) \qquad (n \in \mathbb{N}_0).$$

Notice that

$$\rho_{n+1} = s + (1-\tau) \sum_{j=0}^{n} \tau^j (t-s) < s + (1-\tau) \sum_{n=0}^{\infty} \tau^j (t-s) = t.$$

Deduce inductively

$$\begin{aligned} \phi(\rho_0) &\leq \sum_{l=1}^{k} A_l (\rho_1 - \rho_0)^{-\alpha_l} + \varepsilon \phi(\rho_1) \\ &\leq \sum_{j=0}^{n-1} \varepsilon^j \sum_{l=1}^{k} A_l (\rho_{j+1} - \rho_j)^{-\alpha_l} + \varepsilon^n \phi(\rho_n). \end{aligned}$$

Rearranging the sums on the right-hand side and using that $\rho_{j+1} - \rho_j = (1-\tau)\tau^j (t-s)$ yields

$$\sum_{j=0}^{n-1} \varepsilon^j \sum_{l=1}^{k} A_l (\rho_{j+1} - \rho_j)^{-\alpha_l} = \sum_{l=1}^{k} A_l (1-\tau)^{-\alpha_l} (t-s)^{-\alpha_l} \sum_{j=0}^{n-1} \varepsilon^j \tau^{-j\alpha_l}.$$

Choose τ such that $\varepsilon \tau^{-\max_i\{\alpha_i\}} < 1$ and let $n \to \infty$ to conclude

$$\phi(s) \leq (1-\tau)^{-\max_i\{\alpha_i\}} \sum_{j=0}^{\infty} \Big(\varepsilon \tau^{-\max_i\{\alpha_i\}} \Big)^j \sum_{l=1}^{k} A_l (t-s)^{-\alpha_l}. \qquad \square$$

Now, we are ready to conclude the proof of Caccioppoli's inequality with the sole L^2-norm of u on the right-hand side. For the reduction of the differentiability on the right-hand side of the inequality in Lemma 7.3.1, recall that by Gagliardo–Nirenberg's inequality, one can estimate

$$\|\nabla^{m-1} u\|_{\mathrm{L}^2} \leq C \|u\|_{\mathrm{L}^2}^{\theta} \|u\|_{\mathrm{W}^{m,2}}^{1-\theta},$$

for some $\theta \in (0, 1)$. The term involving $\|\nabla^m u\|_{\mathrm{L}^2}$ in the norm of $\|u\|_{\mathrm{W}^{m,2}}$ can then be controlled by means of the first part of Caccioppoli's inequality, so that only terms of differentiability strictly less than m occur on the right-hand side. Using Young's inequality, we can produce an ε in front of the L^2-norm of $\nabla^{m-1} u$ on the right-hand side. This leads to the situation of Lemma 7.3.2.

Due to the implicit dependence of the constants in the Gagliardo–Nirenberg inequality, we have to restrict the size of the parameter s to be away from zero.

Lemma 7.3.3 (Caccioppoli's inequality part 2)**.** *If in the situation of Lemma 7.3.1 also* $0 < 1/2 \leq s < t \leq 1$ *holds, then there exists a constant* $C > 0$ *depending only on* d, N, m, κ, M, θ, ω, *and* Λ *such that*

$$\int_{\mathcal{B}_{sr} \cap \Omega} |\nabla^k u|^2 \, \mathrm{d}x \leq \frac{C}{[(t-s)r]^{2k}} \int_{\mathcal{B}_{tr} \cap \Omega} |u|^2 \, \mathrm{d}x \qquad (1 \leq k \leq m)$$

holds.

Proof. We will prove the following claim by induction on k. Note that the initial step of this induction, i.e., $k = m$, is Lemma 7.3.1 and that we will successively reduce the value of k.
Claim: There exists a constant $C > 0$ depending at most on d, m, κ, N, M, θ, ω, and Λ, such that for all $1/2 \leq s < t \leq 1$

$$\int_{\mathcal{B}_{sr} \cap \Omega} |\nabla^k u|^2 \, \mathrm{d}x \leq C \sum_{l=0}^{k-1} [(t-s)r]^{-2(k-l)} \int_{\mathcal{B}_{tr} \cap \Omega} |\nabla^l u|^2 \, \mathrm{d}x.$$

First of all, we establish Gagliardo–Nirenberg's inequalities on the sets $\mathcal{B}_{\alpha r} \cap \Omega$ with $\alpha \in [1/2, 1]$. As the constant of this inequality may depend on the size of the underlying sets, we rescale the whole situation.

Rescale the function u as $u_{\alpha r} : \frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega) \to \mathbb{C}^N, x \mapsto u(\alpha r x)$ and recall the cases presented in Lemma 7.3.1. Having a closer look onto $\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega)$, we see that in the first case this is just the ball $B([\alpha r]^{-1} x_0, 1)$, which is a Sobolev extension domain of arbitrary order. In the third case, the set $\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega)$ is simply $B([\alpha r]^{-1} x_0, 1) \cap \frac{1}{\alpha r}\Omega$. The radius r is chosen such that $B(x_0, r)$ only hits Dirichlet boundary so that $u_{\alpha r}$ can be identified with its extension by zero to $B([\alpha r]^{-1} x_0, 1)$. As above $B([\alpha r]^{-1} x_0, 1)$ is a Sobolev extension domain of all orders. Case (2) is more interesting. Here, we have for some $x \in \overline{\partial\Omega \setminus D}$

$$\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega) = \Phi_{x,\alpha r}^{-1}(Q([\alpha r]^{-1}\Phi_x(x_0), 1) \cap [\mathbb{R}^{d-1} \times (0, \infty)]),$$

where $\Phi_{x,\alpha r}^{-1}$ is given by $\frac{1}{\alpha r}\Phi_x^{-1}(\alpha r \cdot)$. Note that $\Phi_{x,\alpha r}^{-1}$ is a bi-Lipschitz homeomorphism, with bi-Lipschitz constant bounded by M and that the bisected cube $Q([\alpha r]^{-1}\Phi_x(x_0), 1) \cap [\mathbb{R}^{d-1} \times (0, \infty)]$ is an (ε, δ)-domain in the sense of Jones [56], see Egert [24, Lem. 2.2.20]. Moreover, a global bi-Lipschitz image of an (ε, δ)-domain is again an (ε, δ)-domain so that in our case, ε and δ only depend on d and M. Finally, by Remark 7.1.2 (3)

$$\operatorname{diam}(\Phi_{x,\alpha r}^{-1}(Q([\alpha r]^{-1}\Phi_x(x_0), 1) \cap [\mathbb{R}^{d-1} \times (0, \infty)])) \geq \frac{1}{M\sqrt{d}}$$

so that the extension result of Rogers [82, Thm. 8] yields a Sobolev extension operator of arbitrary order with an operator norm depending only on d and M. Extending in all three cases functions on $\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega)$ to $\mathbb{R}^d$ and using the Gagliardo–Nirenberg inequalities on the whole space, see Nirenberg [78, p. 125], proves that these inequalities hold true on $\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega)$ where the constant solely depends on d, M, and the Gagliardo–Nirenberg constants on $\mathbb{R}^d$. Especially, we find for $\vartheta = \frac{1}{k+1}$

$$\begin{aligned} &\|\nabla^k u_{\alpha r}\|^2_{\mathrm{L}^2(\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega; \mathbb{C}^{Nd^k}))} \\ &\qquad \leq C \|u_{\alpha r}\|^{2\vartheta}_{\mathrm{L}^2(\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega; \mathbb{C}^N))} \Bigg[\sum_{l=0}^{k+1} \int_{\frac{1}{\alpha r}(\mathcal{B}_{\alpha r} \cap \Omega)} |\nabla^l u_{\alpha r}|^2 \, \mathrm{d}x \Bigg]^{1-\vartheta}. \end{aligned}$$

By linear transformation, we find

$$\begin{aligned} &\|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{\alpha r} \cap \Omega; \mathbb{C}^{Nd^k})} \\ &\leq C \Big[(\alpha r)^{-k} \|u\|_{\mathrm{L}^2(\mathcal{B}_{\alpha r} \cap \Omega; \mathbb{C}^N)} \Big]^{2\vartheta} \Bigg[\sum_{l=0}^{k+1} (\alpha r)^{2(l-k)} \int_{\mathcal{B}_{\alpha r} \cap \Omega} |\nabla^l u|^2 \, \mathrm{d}x \Bigg]^{1-\vartheta}. \end{aligned} \tag{7.3}$$

Next, let $1/2 \leq s < \tau \leq t \leq 1$ and apply (7.3) on $\mathcal{B}_{sr} \cap \Omega$ as well as the induction hypothesis to the term involving $\nabla^{k+1} u$, so that

$$
\begin{aligned}
&\|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^{Nd^k})} \\
&\quad \leq C\big[(sr)^{-k}\|u\|_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^N)}\big]^{2\vartheta} \\
&\qquad \cdot \Bigg[\sum_{l=0}^{k} \Big[s^{2(l-k)} + s^2(\tau - s)^{-2(k+1-l)}\Big] r^{2(l-k)} \int_{\mathcal{B}_{\tau r}\cap\Omega} |\nabla^l u|^2 \,\mathrm{d}x\Bigg]^{1-\vartheta}.
\end{aligned}
$$

For some $\delta > 0$, use Young's inequality $ab \leq \vartheta\delta^{(\vartheta-1)/\vartheta} a^{\frac{1}{\vartheta}} + (1-\vartheta)\delta b^{\frac{1}{1-\vartheta}}$, to obtain

$$
\begin{aligned}
&\leq \frac{C\delta^{\frac{\vartheta-1}{\vartheta}}}{(sr)^{2k}} \|u\|^2_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^N)} \\
&\quad + C(1-\vartheta)\delta \sum_{l=0}^{k} \Big[s^{2(l-k)} + s^2(\tau - s)^{-2(k+1-l)}\Big] r^{2(l-k)} \int_{\mathcal{B}_{\tau r}\cap\Omega} |\nabla^l u|^2 \,\mathrm{d}x.
\end{aligned}
$$

Next, choose δ subject to the condition

$$
C\delta \frac{s^2}{(\tau - s)^2} = \frac{1}{8} \quad \Leftrightarrow \quad \delta = \frac{(\tau - s)^2}{8Cs^2}.
$$

Note that $\delta \leq 1/(2C)$, since $s \geq 1/2$ and $\tau - s \leq 1$. Thus,

$$
C\delta\Big[s^{2(l-k)} + s^2(\tau - s)^{-2(k+1-l)}\Big] \leq \frac{s^{2(l-k)}}{2} + \frac{(\tau - s)^{-2(k-l)}}{8}
$$

and, by means of the choice $\vartheta = \frac{1}{k+1}$,

$$
\frac{C\delta^{\frac{\vartheta-1}{\vartheta}}}{s^{2k}} = \frac{C}{s^{2k}\delta^k} = \frac{8^k C^{1+k}}{(\tau - s)^{2k}}.
$$

Returning to the estimate of $\nabla^k u$, estimate each s from below by $\tau - s$ and use for all terms on the right-hand side of differentiability strictly less than k, that $\mathcal{B}_{sr} \cap \Omega$ and $\mathcal{B}_{\tau r} \cap \Omega$ are contained in $\mathcal{B}_{tr} \cap \Omega$. Put everything together to deduce

$$
\begin{aligned}
\|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^{Nd^k})} &\leq \Big[8^k C^{1+k} + \frac{5(1-\vartheta)}{8}\Big] \frac{1}{[(\tau - s)r]^{2k}} \|u\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^N)} \\
&\quad + \frac{5(1-\vartheta)}{8} \sum_{l=1}^{k-1} \frac{1}{[(\tau - s)r]^{2(k-l)}} \|\nabla^l u\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^l})} \\
&\quad + \frac{5(1-\vartheta)}{8} \|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{\tau r}\cap\Omega;\mathbb{C}^{Nd^k})}.
\end{aligned}
$$

Now, we can appeal to Lemma 7.3.2 by means of the following definitions. Let $s_0 := 1/2$, $t_0 := t$, $\alpha_l := 2(k-(l-1))$,

$$\begin{aligned} A_1 &:= \Big[8^k C^{1+k} + \frac{5(1-\vartheta)}{8}\Big] \|u\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^N)}, \\ A_l &:= \frac{5(1-\vartheta)}{8} \|\nabla^{l-1} u\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^{l-1}})} \qquad (2 \le l \le k), \end{aligned}$$

and

$$\phi(s) := \|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^{Nd^k})}.$$

It follows that there exists a constant $C > 0$ (different from the one above but independent of s, τ, r, and t) such that for all $s \le \tau \le t$

$$\|\nabla^k u\|^2_{\mathrm{L}^2(\mathcal{B}_{sr}\cap\Omega;\mathbb{C}^{Nd^k})} \le C \sum_{l=0}^{k-1} \frac{1}{[(\tau - s)r]^{2(k-l)}} \|\nabla^l u\|^2_{\mathrm{L}^2(\mathcal{B}_{tr}\cap\Omega;\mathbb{C}^{Nd^l})}.$$

In particular, this holds true for $\tau = t$, so that we conclude the induction step. $\square$

The following lemma is a vector-valued and local version of the Sobolev embedding theorem, mentioned in the introduction of this chapter.

Lemma 7.3.4. *Let $p \in [1,\infty)$, $n_0 \in \mathbb{N}$, and $(u_n)_{n=1}^{n_0} \subset \mathrm{W}^{1,p}_{\mathbb{D}}(\Omega;\mathbb{C}^N)$. Let $x_0 \in \overline{\Omega}$ and $0 < r \le 1/(4M\sqrt{d})$ be such that either $B(x_0,r) \subset \Omega$ or $x_0 \in \partial\Omega$. If $q \in [1,\infty)$ is such that $0 \le \delta := \frac{1}{p} - \frac{1}{q} \le \frac{1}{d}$, then there exists a constant $C > 0$ depending only on p, q, d, N, and M such that*

$$\begin{aligned} &\Big(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2\Big]^{\frac{q}{2}} \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ &\quad \le C\Big\{ r\Big(\frac{1}{r^d}\int_{\mathcal{B}\cap\Omega} \Big[\sum_{n=1}^{n_0} |\nabla u_n|^2\Big]^{\frac{p}{2}} \,\mathrm{d}x\Big)^{\frac{1}{p}} + \frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2\Big]^{\frac{1}{2}} \,\mathrm{d}x \Big\} \end{aligned}$$

holds, where $\mathcal{B} = B(x_0, M^2\sqrt{d}r)$ if $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \le 1/(2M)$, and $\mathcal{B} = B(x_0,r)$ else.

Proof. If $x_0 \in \partial\Omega$, extend a function $u \in \mathrm{W}^{1,2}_{\mathbb{D}}(\Omega;\mathbb{C}^N)$ from $\Omega \cap B(x_0,r)$ to $B(x_0,r)$ componentwise as follows: If $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$, then $B(x_0,r) \subset U_{x_0,M\sqrt{d}r}$ by Remark 7.1.2 (3), so define $(\mathcal{E}u)_i := \mathcal{E}_{x_0,M\sqrt{d}r} u_i$ by using the local extension operator $\mathcal{E}_{x_0,M\sqrt{d}r}$ given by Proposition 7.1.3. For all other $x_0 \in \partial\Omega$, let $\mathcal{E}u$ denote the extension by zero.

We present the most difficult case where $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$ and point out changes in the proof for the other cases afterwards. Note that as $U^+_{x_0,r/(M\sqrt{d})}$ is the bi-Lipschitz image of a set with measure comparable to r^d, it holds

$$|U^+_{x_0,r/(M\sqrt{d})}| \geq \mathcal{C} r^d \tag{7.4}$$

with $\mathcal{C}$ depending only on d and M. By Remark 7.1.2 (3), we conclude that $|B(x_0,r) \cap \Omega| \geq \mathcal{C} r^d$ holds true. For a function g, denote its average on $B(x_0,r) \cap \Omega$ by $(g)_{B(x_0,r)\cap\Omega}$. Use the triangle inequality to conclude

$$\begin{aligned}
&\Big(\int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2 \Big]^{\frac{q}{2}} \, \mathrm{d}x \Big)^{\frac{1}{q}} \\
&\qquad \leq \Big(\int_{B(x_0,r)} \Big[\sum_{n=1}^{n_0} \Big| \mathcal{E}u_n(x) - (\mathcal{E}u_n)_{B(x_0,r)\cap\Omega} \Big|^2 \Big]^{\frac{q}{2}} \, \mathrm{d}x \Big)^{\frac{1}{q}} \\
&\qquad\qquad + [|B(0,1)|\, r^d]^{\frac{1}{q}} \Big[\sum_{n=1}^{n_0} \Big| (u_n)_{B(x_0,r)\cap\Omega} \Big|^2 \Big]^{\frac{1}{2}}.
\end{aligned}$$

Next, Gilbarg and Trudinger [43, Lem. 7.16] together with (7.4), show that $|\mathcal{E}u_n(x) - (\mathcal{E}u_n)_{B(x_0,r)\cap\Omega}| \leq 2^d/(d\mathcal{C}) \int_{B(x_0,r)} |x-y|^{1-d} \, |\nabla \mathcal{E}u_n(y)| \, \mathrm{d}y$. Apply this to the first term and apply Minkowski's inequality as well as (7.4) to the second term on the right-hand side, to obtain

$$\begin{aligned}
&\leq \frac{2^d}{d\mathcal{C}} \Big\| \int_{B(x_0,r)} |\cdot - y|^{1-d} \Big[\sum_{n=1}^{n_0} |\nabla \mathcal{E}u_n(y)|^2 \Big]^{\frac{1}{2}} \, \mathrm{d}y \Big\|_{\mathrm{L}^q(B(x_0,r))} \\
&\qquad + \frac{|B(0,1)|^{\frac{1}{q}} \, r^{\frac{d}{q}-d}}{\mathcal{C}} \int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2 \Big]^{\frac{1}{2}} \, \mathrm{d}x.
\end{aligned}$$

For the first term on the right-hand side use the boundedness of the Riesz potential, see [43, Lem. 7.12] for $\delta < 1/d$ and Adams and Hedberg [2, Thm. 3.1.4 (b)] for $\delta = 1/d$, to get

$$\leq \frac{2^d}{d\mathcal{C}} B r^{1-d\delta} \Big\| \Big[\sum_{n=1}^{n_0} |\nabla \mathcal{E}u_n|^2 \Big]^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(B(x_0,r))}$$

$$+\frac{|B(0,1)|^{\frac{1}{q}}\, r^{\frac{d}{q}-d}}{\mathcal{C}} \int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2\Big]^{\frac{1}{2}} \, \mathrm{d}x,$$

where B depends only on d, p, and q. Note that Proposition 7.1.3 gives

$$\begin{aligned} \Big\| \Big[\sum_{n=1}^{n_0} |\nabla \mathcal{E} u_n|^2\Big]^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(B(x_0,r)\setminus\Omega)} &= \Big\| \Big[\sum_{n=1}^{n_0} |\nabla [u_n \circ \psi]|^2\Big]^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(B(x_0,r)\setminus\Omega)} \\ &\leq C \Big\| \Big[\sum_{n=1}^{n_0} |\nabla u_n|^2\Big]^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(U^+_{x_0,M\sqrt{d}r})}, \end{aligned}$$

where $C > 0$ depends only on d and M. By Remark 7.1.2 (3), we conclude the proof for $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$.

If $B(x_0, r) \subset \Omega$, we do exactly the same without the extension operator. If $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) > 1/(2M)$, we proceed as above with the one exception that in the first inequality below (7.4), we introduce $(\mathcal{E}u_n)_{B(x_0,r)}$ instead of $(\mathcal{E}u_n)_{B(x_0,r)\cap\Omega}$. This has the effect of avoiding the need of an estimate of the form $|B(x_0, r)\cap\Omega| \geq \mathcal{C}r^d$, which is not available under the given geometric setup. $\square$

The following lemma is an iterated version of the previous one.

Lemma 7.3.5. *Let $p \in [1, \infty)$, $n_0 \in \mathbb{N}$, and $(u_n)_{n=1}^{n_0} \subset \mathrm{W}^{m,p}_{\mathbb{D}}(\Omega; \mathbb{C}^N)$. Let $x_0 \in \overline{\Omega}$ and $0 < r \leq [M^2\sqrt{d}]^{1-m}/(4M\sqrt{d})$ be such that either $B(x_0, r) \subset \Omega$ or $x_0 \in \partial\Omega$. If $q \in [1, \infty)$ satisfies $0 \leq \delta := \frac{1}{p} - \frac{1}{q} \leq \frac{m}{d}$, then there exists a constant $C > 0$ depending at most on p, q, d, N, m, and M such that*

$$\begin{aligned} &\Big(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} |u_n|^2\Big]^{\frac{q}{2}} \, \mathrm{d}x\Big)^{\frac{1}{q}} \\ &\qquad \leq C\Big\{ r^m \Big(\frac{1}{r^d} \int_{\mathcal{B}_m\cap\Omega} \Big[\sum_{n=1}^{n_0} |\nabla^m u_n|^2\Big]^{\frac{p}{2}} \, \mathrm{d}x\Big)^{\frac{1}{p}} \\ &\qquad\qquad + \sum_{k=0}^{m-1} \frac{1}{r^{d-k}} \int_{\mathcal{B}_k\cap\Omega} \Big[\sum_{n=1}^{n_0} |\nabla^k u_n|^2\Big]^{\frac{1}{2}} \, \mathrm{d}x \Big\} \end{aligned}$$

holds, where $\mathcal{B}_k := B(x_0, [M^2\sqrt{d}]^k r)$ if $x_0 \in \partial\Omega$ with $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$ and $\mathcal{B}_k := B(x_0, r)$ else.

Proof. We will only consider the case where $x_0 \in \partial\Omega$ satisfies the inequality $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$. The other cases are proven in the same way, with the one exception, that the domain of integration stays the same in each iteration step. Define $p_1 \in [p, \infty)$ via the equation

$$\frac{1}{p} - \frac{1}{p_1} = \frac{\delta}{m}.$$

It follows that

$$\frac{1}{p_1} = \frac{1}{p} - \frac{\delta}{m} \geq \frac{1}{p} - \frac{1}{p} + \frac{1}{q},$$

so that in fact $p_1 \in [p, q]$. Inductively, define for $2 \leq k \leq m$ the number $p_k \in [p_{k-1}, q]$ via

$$\frac{1}{p_{k-1}} - \frac{1}{p_k} = \frac{\delta}{m}.$$

In particular, one finds $p_m = q$, so that Lemma 7.3.4 can be applied iteratively. Thus,

$$\begin{aligned}
&\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega}\left[\sum_{n=1}^{n_0}|u_n|^2\right]^{\frac{q}{2}}\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\quad\leq C\Bigg\{r\left(\frac{1}{r^d}\int_{\mathcal{B}_1\cap\Omega}\left[\sum_{n=1}^{n_0}|\nabla u_n|^2\right]^{\frac{p_{m-1}}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_{m-1}}} \\
&\qquad+\frac{1}{r^d}\int_{\mathcal{B}_0\cap\Omega}\left[\sum_{n=1}^{n_0}|u_n|^2\right]^{\frac{1}{2}}\,\mathrm{d}x\Bigg\} \\
&\quad\leq C\Bigg\{r^2\left(\frac{1}{r^d}\int_{\mathcal{B}_2\cap\Omega}\left[\sum_{n=1}^{n_0}|\nabla^2 u_n|^2\right]^{\frac{p_{m-2}}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_{m-2}}} \\
&\qquad+\frac{1}{r^{d-1}}\int_{\mathcal{B}_1\cap\Omega}\left[\sum_{n=1}^{n_0}|\nabla u_n|^2\right]^{\frac{1}{2}}\,\mathrm{d}x+\frac{1}{r^d}\int_{\mathcal{B}_0\cap\Omega}\left[\sum_{n=1}^{n_0}|u_n|^2\right]^{\frac{1}{2}}\,\mathrm{d}x\Bigg\}.
\end{aligned}$$

Inductively, it follows

$$\begin{aligned}
&\leq C\Bigg\{r^m\left(\frac{1}{r^d}\int_{\mathcal{B}_m\cap\Omega}\left[\sum_{n=1}^{n_0}|\nabla^m u_n|^2\right]^{\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\qquad+\sum_{k=0}^{m-1}\frac{1}{r^{d-k}}\int_{\mathcal{B}_k\cap\Omega}\left[\sum_{n=1}^{n_0}|\nabla^k u_n|^2\right]^{\frac{1}{2}}\,\mathrm{d}x\Bigg\}.
\end{aligned}$$

□

Now we are in the position to prove the vector-valued weak reverse Hölder estimates.

Theorem 7.3.6. *Let $n_0 \in \mathbb{N}$, $(\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_{\pi-\theta}$, $x_0 \in \overline{\Omega}$, and $0 < r \leq 1/(8[M^2\sqrt{d}]^{m+1})$ be such that either $B(x_0, 3M[M^2\sqrt{d}]^{m+1}r) \subset \Omega$ or $x_0 \in \partial\Omega$. Moreover, let $(f_n)_{n=1}^{n_0} \subset \mathrm{L}^2(\Omega;\mathbb{C}^N)$ such that for every $1 \leq n \leq n_0$ the function f_n vanishes on $B(x_0, 3M[M^2\sqrt{d}]^{m+1}r) \cap \Omega$. If $2m \geq d$, let q be any number larger than 2 and if $2m < d$, let $q = \frac{2d}{d-2m}$. Then there exists a constant $C > 0$, depending at most on d, N, q, κ, m, M, θ, ω, and Λ such that*

$$\Big(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega}\Big[\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\Big]^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{1}{q}} \leq C\Big(\frac{1}{r^d}\int_{B(x_0,2[M^2\sqrt{d}]^{m+1}r)\cap\Omega}\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\,\mathrm{d}x\Big)^{\frac{1}{2}}$$

holds.

Proof. Apply Lemma 7.3.5 with q and $p = 2$ to the functions $(\lambda_n u_n)_{n=1}^{n_0}$ to get

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega}\Big[\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\Big]^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{1}{q}} &\leq C\Big\{r^m\Big(\frac{1}{r^d}\int_{B(x_0,[M^2\sqrt{d}]^m r)\cap\Omega}\sum_{n=1}^{n_0}[|\lambda_n|\,|\nabla^m u_n|]^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\quad+\sum_{k=0}^{m-1}\frac{1}{r^{d-k}}\int_{B(x_0,[M^2\sqrt{d}]^k r)\cap\Omega}\Big[\sum_{n=1}^{n_0}[|\lambda_n|\,|\nabla^k u_n|]^2\Big]^{\frac{1}{2}}\,\mathrm{d}x\Big\}.\end{aligned}$$

Apply Hölder's inequality to the second term on the right-hand side, so that with a different constant C

$$\leq C\sum_{k=0}^{m} r^k\Big(\frac{1}{r^d}\int_{B(x_0,[M^2\sqrt{d}]^k r)\cap\Omega}\sum_{n=1}^{n_0}[|\lambda_n|\,|\nabla^k u_n|]^2\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

If x_0 is either inside Ω or on $\partial\Omega$ with $\mathrm{dist}(x_0, \overline{\partial\Omega\setminus D}) > 1/(2M)$, then $2[M^2\sqrt{d}]^m r \leq 1/(4M^2\sqrt{d})$, so that Lemma 7.3.3 is directly applicable with $s = 1/2$, $t = 1$, and radius $2[M^2\sqrt{d}]^k r$. This yields

$$\leq C\Big(\frac{1}{r^d}\int_{B(x_0,2[M^2\sqrt{d}]^m r)\cap\Omega}\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

In order to employ Caccioppoli's inequality in the case $\operatorname{dist}(x_0, \overline{\partial\Omega \setminus D}) \leq 1/(2M)$, apply Remark 7.1.2 (3), which in the current situation reads as

$$B(x_0, [M^2\sqrt{d}]^k r) \subset U_{x_0, M\sqrt{d}[M^2\sqrt{d}]^k r} \subset B(x_0, [M^2\sqrt{d}]^{k+1} r).$$

Since $2[M^2\sqrt{d}]^{m+1} r \leq 1/4$, Caccioppoli's inequality is applicable on the sets $\Omega \cap U_{x_0, 2[M^2\sqrt{d}]^{k+1} r}$ with $s = 1/2$ and $t = 1$. Proceeding as above concludes the proof. □

7.3.1 Proof of Theorem 7.2.4

By Remark 7.2.5 it suffices to concentrate on the case $p > 2$. Thus, in the case $2m < d$, define $p := 2d/(d-2m)$, and in the case $2m \geq d$, let $p > 2$ be arbitrary.

As discussed in Section 7.2, the L^2-realization of the elliptic operator A is sectorial of angle $\omega \in [0, \frac{\pi}{2})$. Thus, for every $\theta \in (\omega, \pi]$ the family $\{\lambda(\lambda + A)^{-1}\}_{\lambda \in \mathrm{S}_{\pi-\theta}}$ is bounded, which is equivalent to the boundedness of the family $\mathcal{T}$ given by

$$\{(\lambda_1(\lambda_1 + A)^{-1}, \dots, \lambda_{n_0}(\lambda_{n_0} + A)^{-1}, 0, \dots) : n_0 \in \mathbb{N}, (\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_{\pi-\theta}\}$$

on $\mathrm{L}^2(\Omega; \ell^2(\mathbb{C}^N))$. With regard to Theorem 3.1.2, fix $X = Y = \ell^2(\mathbb{C}^N)$ and invoke Remark 3.1.3 and Proposition 2.3.4 to conclude that for each operator $T \in \mathcal{T}$ one has to verify the weak reverse Hölder estimates with uniform constants. Thus, we have to show that there exist constants $C > 0$, $R_0 > 0$, $\alpha_2 > \alpha_2 > 1$ such that for all $n_0 \in \mathbb{N}$, $(\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_{\pi-\theta}$, $(f_n)_{n=1}^{n_0} \subset \mathrm{L}^\infty(\Omega; \mathbb{C}^N)$ with $f_n = 0$ on $B(x_0, \alpha_2 r) \cap \Omega$, and $u_n := (\lambda_n + A)^{-1} f_n$ the estimate

$$\begin{aligned}\bigg(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} \Big[\sum_{n=1}^{n_0} [|\lambda_n|\,|u_n|]^2\Big]^{\frac{p}{2}} \,\mathrm{d}x\bigg)^{\frac{1}{p}} \\ \leq C \bigg(\frac{1}{r^d} \int_{B(x_0,\alpha_1 r)\cap\Omega} \sum_{n=1}^{n_0} [|\lambda_n|\,|u_n|]^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}}\end{aligned}$$

holds. This is exactly the statement of Theorem 7.3.6. Furthermore, Lemma 3.2.2 shows that one can take $\alpha_1 = 2$ and that the weak reverse Hölder estimates are valid for every x_0 satisfying $B(x_0, r) \cap \Omega \neq \emptyset$. If

$2m < d$, this allows us to invoke the self-improving property of weak reverse Hölder estimates, see Proposition 3.1.4. Hence, there exists $\varepsilon > 0$ such that

$$\Big(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega}\Big[\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\Big]^{\frac{p+\varepsilon}{2}}\,\mathrm{d}x\Big)^{\frac{1}{p+\varepsilon}}$$
$$\leq C\Big(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega}\Big[\sum_{n=1}^{n_0}[|\lambda_n|\,|u_n|]^2\Big]^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

We conclude that $\{\lambda(\lambda + A_p)^{-1}\}_{\lambda\in\mathrm{S}_{\pi-\theta}}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\mathrm{L}^p(\Omega;\mathbb{C}^N))$. By Remark 2.1.2 we infer that A_p is densely defined and by virtue of Theorem 2.3.5, that A_p has maximal L^q-regularity. $\square$

Bibliography

[1] R. A. Adams and J. J. F. Fournier. Sobolev Spaces. Pure and Applied Mathematics, vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] D. R. Adams and L. I. Hedberg. Function spaces and potential theory. Grundlehren der mathematischen Wissenschaften, vol. 314. Springer, Berlin, 1996.

[3] H. Amann. Linear and quasilinear parabolic problems. Monographs in Mathematics, vol. 89, Birkhäuser, Boston, 1995.

[4] H. Amann and J. Escher. Analysis III. Grundstudium Mathematik. Birkhäuser, Basel-Boston-Berlin, 2008.

[5] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. Vector-valued Laplace Transforms and Cauchy Problems. Monographs in Mathematics, vol. 96, Birkhäuser, Basel-Boston-Berlin, 2001.

[6] P. Auscher. *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates.* Mem. Amer. Math. Soc. **186** (2007), no. 871.

[7] A. Barton. *Elliptic partial differential equations with almost-real coefficients.* Mem. Amer. Math. Soc. **223** (2013), no. 1051.

[8] A. Barton. *Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients.* Available at `http://arxiv.org/abs/1409.7600v1`

[9] J. Bergh and J. Löfström. Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften, no. 223, Springer, Berlin, 1976.

[10] P. Billingsley. Probability and measure. Third edition. John Wiley & Sons, New York, 1995.

[11] M. E. Bogovskiĭ. *Solutions of some problems of vector analysis, associated with the operators* div *and* grad. Trudy Sem. S. L. Soboleva (1980), no. 1, 5–40.

[12] K. Brewster, D. Mitrea, I. Mitrea, and M. Mitrea. *Extending Sobolev functions with partially vanishing traces from locally (ε, δ)-domains and applications to mixed boundary problems.* J. Funct. Anal. **266** (2014), no. 7, 4314–4421.

[13] R. M. Brown and Z. Shen. *Estimates for the Stokes operator in Lipschitz domains.* Indiana Univ. Math. J. **44** (1995), no. 4, 1183–1206.

[14] L. A. Caffarelli and I. Peral. *On $W^{1,p}$ estimates for elliptic equations in divergence form.* Comm. Pure Appl. Math **51** (1998), no. 1, 1–21.

[15] A. P. Calderón and A. Torchinsky. *Parabolic maximal functions associated with a distribution.* Advances in Math. **16** (1975), 1–64.

[16] L. Cattabriga. *Su un problema al contorno relativo al sistema di equazioni di Stokes.* Rend. Sem. Mat. Univ. Padova **31** (1961), 308–340.

[17] H. J. Choe and H. Kozono. *The Stokes problem for Lipschitz domains.* Indiana Univ. Math. J. **51** (2002), no. 5, 1235–1260.

[18] R. Denk, M. Hieber, and J. Prüss. *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type.* Mem. Amer. Math. Soc. **166** (2003), no. 788.

[19] R. Denk and M. Kaip. General Parabolic Mixed Order Systems in L_p and Applications. Operator Theory: Advances and Applications, vol. 239. Birkhäuser/Springer, Cham, 2013.

[20] R. Denk, J. Saal, and J. Seiler. *Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity*. Russian J. Math. Phys. **15** (2008), no. 2, 171–192.

[21] P. Deuring. *The Stokes resolvent in 3D domains with conical boundary points: nonregularity in L^p-spaces*. Adv. Differential Equations **6** (2001), no. 2, 175–228.

[22] P. Deuring and W. von Wahl. *Strong solutions of the Navier-Stokes system in Lipschitz bounded domains*. Math. Nachr. **171** (1995), 111–148.

[23] J. Diestel, H. Jarchow, and A. Tonge. Absolutely summing operators. Cambridge Studies in Advanced Mathematics, vol. 43. Cambridge University Press, Cambridge, 1995.

[24] M. Egert. On Kato's conjecture and mixed boundary conditions. Thesis, Sierke Verlag, Göttingen, 2016.

[25] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, vol. 194. Springer, New York, 2000.

[26] L. Escauriaza, G. Seregin, and V. Šveràk. *Backward uniqueness for parabolic equations*. Arch. Ration. Mech. Anal. **169** (2003), no. 2, 147–157.

[27] L. C. Evans and R. F. Gariepy. Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, CRC Press, Boca Raton FL, 1992.

[28] E. B. Fabes, C. E. Kenig, and G. C. Verchota. *The Dirichlet problem for the Stokes system on Lipschitz domains*. Duke Math. J. **57** (1988), no. 3, 769–793.

[29] E. Fabes, O. Mendez, and M. Mitrea. *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains* . J. Funct. Anal. **159** (1998), no. 2, 323–368.

[30] R. Farwig and H. Sohr. *On the existence of local strong solutions for the Navier-Stokes equations in completely general domains*. Nonlinear Anal. **73** (2010), no. 6, 1459–1465.

[31] H. Federer. Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer, New York, 1969.

[32] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equations. The millenium prize problems. Clay Math. Inst., Cambridge, 2006.

[33] D. Fujiwara. *On the asymptotic behaviour of the Green operators for elliptic boundary value problems and the pure imaginary powers of some second order operators.* J. Math. Soc. Japan **21** (1969), 481–521.

[34] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Springer Monographs in Mathematics. Springer, New York, 2011.

[35] M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis. *Maximal $L^p - L^q$-Estimates for the Stokes Equation: a Short Proof of Solonnikov's Theorem.* J. math. fluid mech. **12** (2010), 47–60.

[36] J. Geng and J. Kilty. *The L^p regularity problem for the Stokes system on Lipschitz domains.* J. Differential Equations **259** (2015), no. 4, 1275–1296.

[37] M. Giaquinta and L. Martinazzi. An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs. Scuola Normale Superiore di Pisa (Nuova Serie). Edizioni della Normale, Pisa, 2012.

[38] M. Giaquinta and G. Modica. *Regularity results for some classes of higher order nonlinear elliptic systems.* J. Reine Angew. Math. **311/312** (1979), 145–169.

[39] Y. Giga. *Analyticity of the semigroup generated by the Stokes operator in L_r spaces.* Math. Z. **178** (1981), no. 3, 297–329.

[40] Y. Giga. *Domains of fractional powers of the Stokes operator in L_r spaces.* Arch. Rational Mech. Anal. **89** (1985), no. 3, 251–265.

[41] Y. Giga. *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system.* J. Differential Equations **62** (1986), no. 2, 186–212.

[42] Y. Giga and T. Miyakawa. *Solutions in L_r of the Navier-Stokes Initial Value Problem.* Arch. Rational Mech. Anal. 89 (1984), no. 3, 267–281.

[43] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order. Classics in Mathematics. Springer, Berlin, 2001.

[44] L. Grafakos. Classical Fourier Analysis. Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008.

[45] P. Grisvard. Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, vol. 24. Pitman (Advanced Publishing Program), Boston, 1985.

[46] M. Haase. The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel, 2006.

[47] R. Haller-Dintelmann, A. Jonsson, D. Knees, and J. Rehberg. *On elliptic and parabolic regularity for second order divergence operators with mixed boundary conditions.* Available at `http://arxiv.org/abs/1310.3679`.

[48] P. Hajłasz. *Change of variables formula under minimal assumptions.* Colloq. Math. 64 (1993), no. 1, 93–101.

[49] L. I. Hedberg and T. Kilpeläinen. *On the stability of Sobolev spaces with zero boundary values.* Math. Scand. **85** (1999), no. 2, 245–258.

[50] J. Heinonen. Lectures on Lipschitz Analysis. Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 100, University of Jyväskylä, 2005.

[51] E. Hopf. *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.* Math. Nachr. **4** (1951), 213–231.

[52] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. Analysis in Banach Spaces, 2016, in preparation.

[53] D. Jerison and C. E. Kenig. *The Inhomogeneous Dirichlet Problem in Lipschitz Domains.* J. Funct. Anal. **130** (1995), no. 1, 161–219.

[54] D. Jerison and C. E. Kenig. *The Neumann problem on Lipschitz domains.* Bull. Amer. Math. Soc. **4** (1981), no. 2, 203–207.

[55] H. Jia and V. Šveràk. *Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space?*. J. Funct. Anal. **268** (2015), no. 12, 3734–3766.

[56] P. W. Jones. *Quasiconformal mappings and extendability of functions in Sobolev spaces*. Acta Math. **147** (1981), no. 1-2, 71–88.

[57] N. J. Kalton and L. Weis. *The H^∞-calculus and sums of closed operators*. Math. Ann **312** (2001), no. 2, 319–345.

[58] T. Kato. Perturbation theory for linear operators. Die Grundlehren der mathematischen Wissenschaften, vol. 132. Springer, Berlin, 1995.

[59] T. Kato. *Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions*. Math. Z. **187** (1984), no. 4, 471–480.

[60] T. Kato and H. Fujita. *On the nonstationary Navier-Stokes system*. Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[61] P. C. Kunstmann and L. Weis. *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*. In Functional analytic methods for evolution equations, Lecture Notes in Mathematics, vol. 1855, Springer, Berlin, 2004, 65–311.

[62] O. A. Ladyzhenskaya. The mathematical theory of viscous incompressible flow. Mathematics and its Applications, vol. 2. Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[63] N. N. Lebedev. Special functions and their applications. Translated and edited by Richard A. Silverman. Dover Publications, Inc., New York, 1972.

[64] J. Leray. *Essai sur les mouvements plans d'un liquide visqueux que limitent des parois*. J. Math. Pures Appl. **13** (1934), 331–418.

[65] J. Leray. *Etude de diverses équations intégrales nonlinéaires et quelque problèmes que pose l'hydrodynamique*. J. Math. Pures Appl. **12** (1933), 1–82.

[66] J. Leray. *Sur le mouvement d'un liquide visqueux emplissant l'espace*. Acta Math. **63** (1934), 193–248.

[67] A. Lunardi. Interpolation Theory. Scuola Normale Superiore di Pisa (Nuova Serie), Pisa, 2009.

[68] V. N. Maslennikova and M. E. Bogovskiĭ. *Elliptic boundary value problems in unbounded domains with noncompact and nonsmooth boundaries.* Rend. Sem. Mat. Fis. Milano **56** (1986), 125–138.

[69] V. Maz'ya. Sobolev spaces with applications to elliptic partial differential equations. Die Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer, Heidelberg, 2011.

[70] W. McLean. Strongly elliptic systems and boundary integral equations. Cambridge University Press, Cambridge, 2000.

[71] M. Meyries and M. Veraar. *Sharp embedding results for spaces of smooth functions with power weights.* Studia Math. **208** (2012), no. 3, 257–293.

[72] D. Mitrea. *Sharp L^p-Hodge decompositions for Lipschitz domains in* $\mathbb{R}^2$. Adv. Differential Equations **7** (2002), no. 3, 343–364.

[73] M. Mitrea and S. Monniaux. *On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds.* Trans. Amer. Math. Soc. **361** (2009), no. 6, 3125–3157.

[74] M. Mitrea and S. Monniaux. *The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains.* Differential Integral Equations **22** (2009), no. 3-4, 339–356.

[75] M. Mitrea and S. Monniaux. *The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains.* J. Funct. Anal. **254** (2008), no. 6, 1522–1574.

[76] M. Mitrea, S. Monniaux, and M. Wright. *The Stokes operator with Neumann boundary conditions in Lipschitz domains.* J. Math. Sci. **176** (2011), no. 3, 409–457.

[77] M. Mitrea and M. Wright. *Boundary value problems for the Stokes system in arbitrary Lipschitz domains.* Astérisque **344** (2012).

[78] L. NIRENBERG. *On elliptic partial differential equations.* Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162.

[79] J. NEČAS. Direct methods in the theory of elliptic equations. Monographs in Mathematics. Springer, Heidelberg, 2012.

[80] E. M. OUHABAZ. Analysis of Heat Equations on Domains. London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton NJ, 2005.

[81] J. PIPHER and G. C. VERCHOTA. *Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators.* Ann. of Math. (2) **142** (1995), no. 1, 1–38.

[82] L. G. ROGERS. *Degree-independent Sobolev extension on locally uniform domains.* J. Funct. Anal. **235** (2006), no. 2, 619–665.

[83] W. RUDIN. Real and complex analysis. McGraw-Hill Book Co., New York, 1987.

[84] J. SAAL. *Strong solutions for the Navier-Stokes equations on bounded and unbounded domains with a moving boundary.* Electron. J. Differ. Equ. Conf. **15** (2007), 365–375.

[85] M. SCHECHTER. Principles of functional analysis. Graduate Studies in Mathematics, vol. 36. American Mathematical Society, Providence, 2002.

[86] Z. SHEN. *Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders.* Amer. J. Math. **113** (1991), no. 2, 293–373.

[87] Z. SHEN. *Bounds on Riesz transforms on L^p-spaces for second order elliptic operators.* Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 173–197.

[88] Z. SHEN. *Resolvent estimates in L^p for elliptic systems in Lipschitz domains.* J. Funct. Anal. **133** (1995), no. 1, 224–251.

[89] Z. SHEN. *Resolvent estimates in L^p for the Stokes operator in Lipschitz domains.* Arch. Ration. Mech. Anal. (2012), no. 2, 395–424.

[90] H. SOHR. The Navier-Stokes equations. An elementary functional analytic approach. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2001.

[91] V. A. Solonnikov. *Estimates for solutions of nonstationary Navier-Stokes equations*. J. Soviet Math. **8** (1977), no. 4, 467–529.

[92] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, 1986.

[93] M. E. Taylor. *Incompressible fluid flows on rough domains.* In: Semigroups of operators: theory and applications (Newport Beach, CA, 1998). Progr. Nonlinear Differential Equations Appl., vol. 42, Birkhäuser, Basel, 2000, 320–334.

[94] H. Triebel. Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, vol. 18, North-Holland Publishing, Amsterdam, 1978.

[95] G. C. Verchota. Layer Potentials and Boundary Value Problems for Laplace's Equation on Lipschitz Domains. Thesis, University of Minnesota, 1982.

[96] G. C. Verchota. *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*. J. Funct. Anal. **59** (1984), no. 3, 572–611.

[97] W. Wei and Z. Zhang. L^p *resolvent estimates for constant coefficient elliptic systems on Lipschitz domains.* J. Funct. Anal. **267** (2014), no. 9, 3262–3293.

[98] J. Weidmann. Linear Operators in Hilbert Spaces. Graduate Texts in Mathematics, vol. 68. Springer, New York-Berlin, 1980.

[99] L. Weis. *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*. Math. Ann. **319** (2001), no. 4, 735–758.

[100] M. Wiegner. *The Navier-Stokes equations–a neverending challenge*? Jahresber. Deutsch. Math.-Verein. **101** (1999), no. 1, 1–25.

[101] K. Yosida. Functional analysis. Die Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer, Berlin-New York, 1980.

[102] W. P. Ziemer. Weakly Differentiable Functions. Graduate Texts in Mathematics, vol. 120. Springer, New York, 1989.

List of notations

General

a.e.	short for almost everywhere	
s.t.	short for such that	
p.v.	principle value integral	
δ_{jk}	Kronecker's delta	
$\langle \cdot, \cdot \rangle$	inner product on $\mathbb{C}^l$	

Sets

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$	natural numbers, natural numbers including zero, real numbers, complex numbers	
$\overline{A}$	Closure of a set A	
$\mathrm{dist}(A, B)$	distance of two sets A and B	
$B'(x, r)$	ball in $\mathbb{R}^{d-1}$ centered in x and with radius r	15
$\mathbb{D}$	N-tupel of closed sets $D_1, \ldots, D_N$	6
S_θ	open sector in the complex plane about the positive real axis and with opening angle 2θ	51

Integration

Spaces

Functions and operators

Φ	Canonical isomorphism between $\mathrm{L}^p_\sigma(\Omega)$ and $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ 148
$K(t,x,X,Y)$	K-functional 12
Γ	Γ-function ....
$\mathrm{H}^{(1)}_\nu$	Hankel function of the first kind 86
J_ν	Bessel function of the first kind of order ν 98
Y_ν	Bessel function of the second kind of order ν 98
$\mathrm{G}(x;0)$	Fundamental solution of the Laplacian 84
$\mathrm{G}(x;\lambda)$	Fundamental solution to the scalar Helmholtz equation 86
$\Gamma(x;0)$	Fundamental solution to the Stokes problem 85
$\Gamma(x,\lambda)$	Fundamental solution of the Stokes resolvent problem 88
Φ	Pressure vector 85
$\mathcal{S}_\mathrm{L}$	Single layer potential for the Laplacian 84
$\mathcal{S}_0$	Single layer potential for the Stokes system 85
$\mathcal{S}_\lambda$	Single layer potential for the Stokes resolvent 88
ϕ	Pressure corresponding to the Stokes single layer potentials 85

Boundary value problems

$(\mathrm{Neu}_\mathrm{L})$	L^2-Neumann problem of the Laplacian 82
$(\mathrm{Dir}_{\mathrm{S},\lambda})$	L^2-Dirichlet problem of the Stokes (resolvent) equation 83
$(\mathrm{Reg}_{\mathrm{S},\lambda})$	regularity for the L^2-Dirichlet problem of the Stokes (resolvent) equation 83

Index

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